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Orthonormal polynomials for generalized Freud-type weights and higher-order Hermite–Fejér interpolation polynomials

T. Kasuga[a] and R. Sakai[b,*]

[a] *Department of Mathematics, Kumamoto National College of Technology, Nishigohshi-machi, Kikuchi-gun, Kumamoto 861-1102, Japan*

[b] *Department of Mathematics, Asuke Senior High School, Kawahara 5, Yagami, Asuke-cho, Higashikamo-gun, Aichi 444-2451, Japan*

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Abstract

Let $Q: \mathbf{R} \to \mathbf{R}$ be even, nonnegative and continuous, Q' be continuous, $Q' > 0$ in $(0, \infty)$, and let Q'' be continuous in $(0, \infty)$. Furthermore, Q satisfies further conditions. We consider a certain generalized Freud-type weight $W_{rQ}^2(x) = |x|^{2r} \exp(-2Q(x))$. In previous paper (J. Approx. Theory 121 (2003) 13) we studied the properties of orthonormal polynomials $\{P_n(W_{rQ}^2; x)\}_{n=0}^{\infty}$ with the generalized Freud-type weight $W_{rQ}^2(x)$ on $\mathbf{R}$. In this paper we treat three themes. Firstly, we give an estimate of $P_n(W_{rQ}^2; x)$ in the L_p-space, $0 < p \leqslant \infty$. Secondly, we obtain the Markov inequalities, and third we study the higher-order Hermite–Fejér interpolation polynomials based at the zeros $\{x_{kn}\}_{k=1}^{n}$ of $P_n(W_{rQ}^2; x)$. In Section 5 we show that our results are applicable to the study of approximation for continuous functions by the higher-order Hermite-Fejér interpolation polynomials.

Keywords: Generalized Freud-type weights; Orthonormal polynomials; Markov inequalities; Higher-order Hermite-Fejer interpolation

*Corresponding author.
E-mail addresses: kasuga@ge.knct.ac.jp (T. Kasuga), ryozi@mcc.spacetown.ne.jp (R. Sakai).

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0. Introduction

Let $Q: \mathbf{R} \to \mathbf{R}$ be even, nonnegative and continuous, Q' be continuous, $Q' > 0$ in $(0, \infty)$, and let Q'' be continuous in $(0, \infty)$. Furthermore, Q satisfies the following condition:

$$1 < A \leqslant \{(d/dx)(xQ'(x))\}/Q'(x) \leqslant B, \quad x \in (0, \infty), \tag{0.1}$$

where A and B are constants. Let $v = 1, 2, 3, \ldots$. If $v = 1$, then we assume (0.1). For $v \geqslant 2$ we suppose (0.1) and further that $Q \in C^{(v+1)}(\mathbf{R})$ and

$$0 \leqslant xQ^{(j+1)}(x)/Q^{(j)}(x) \leqslant \tilde{B}, \quad j = 2, 3, \ldots, v,$$

$$Q^{(v+1)}(x) \uparrow (\text{nondecreasing}), \quad x \in (0, \infty), \tag{0.2}$$

where $\tilde{B}$ is a positive constant. Then we consider generalized Freud-type weights $W_{rQ}(x)$ such that

$$W_{rQ}(x) = |x|^r \exp(-Q(x)), \quad x \in \mathbf{R}, \tag{0.3}$$

where $r \geqslant 0$ except for Sections 3 and 4, but in Sections 3 and 4 we suppose $r > -1/2$. We say that the weight $W_{rQ}(x)$ satisfies the condition $C(v)$. For simplicity we write $W_Q(x) = W_{0Q}$. We consider the series of orthonormal polynomials $\{P_n(W_{rQ}^2; x)\}_{n=0}^{\infty}$ with weight (0.3), where $P_n(W_{rQ}^2; x) \in \prod_n$ and $\prod_n$ denotes the class of polynomials of degree $\leqslant n$. The orthonormal polynomials are constructed by

$$\int_{-\infty}^{\infty} P_i(W_{rQ}^2; t) P_j(W_{rQ}^2; t) W_{rQ}^2(t)\, dt = \delta_{ij} \text{ (Kronecker's delta)},$$
$$i, j = 0, 1, 2, \ldots .$$

In previous paper [KaS1] we have investigated some interesting properties of orthonormal polynomials $\{P_n(W_{rQ}^2; x)\}_{n=0}^{\infty}$. In this paper we treat three different themes. Firstly, we give an estimate of $P_n(W_{rQ}^2; x)$ in the L_p-space, $0 < p \leqslant \infty$. Secondly, we obtain the Markov inequalities, and third we study the higher-order Hermite–Fejér interpolation polynomials based at the zeros $\{x_{kn}\}_{k=1}^{n}$, $-\infty < x_{nn} < \cdots < x_{2n} < x_{1n} < \infty$, of $P_n(W_{rQ}^2; x)$. In Section 5 we show that our results are applicable to the study of approximation for continuous functions by the higher-order Hermite–Fejér interpolation polynomials. These are also essential to our next study [KaS2] with respect to a necessary and sufficient condition for a convergence of the higher-order Hermite–Fejér interpolation polynomials.

For $f \in C(\mathbf{R})$ we define the higher-order Hermite–Fejér interpolation polynomial $L_n(v, f; x)$ based at the zeros $\{x_{kn}\}_{k=1}^{n}$ as follows:

$$L_n(v, f; x_{kn}) = f(x_{kn}), \quad k = 1, 2, \ldots, n,$$

$$L_n^{(i)}(v, f; x_{kn}) = 0, \quad k = 1, 2, \ldots, n, \ i = 1, 2, \ldots, v-1. \tag{0.4}$$

$L_n(1, f; x)$ is the Lagrange interpolation polynomial, and $L_n(2, f; x)$ is the ordinary Hermite–Fejér interpolation polynomial. The fundamental polynomials $h_{kn}(v; x) \in \prod_{vn-1}$ for the higher-order Hermite–Fejér interpolation polynomials of

(0.4) are defined as follows:

$$h_{kn}(v;x) = l_{kn}^{v}(x) \sum_{i=0}^{v-1} e_i(v,k,n)(x-x_{kn})^i,$$
$$e_i(v,k,n) \ (0 \leqslant i \leqslant v-1): \text{ real coefficients},$$

$$l_{kn}(x) = \frac{P_n(W_{rQ}^2;x)}{(x-x_{kn})P_n'(W_{rQ}^2;x_{kn})}, \quad k=1,2,\dots,n,$$
$$h_{kn}(v;x_{pn}) = \delta_{kp}, \quad h_{kn}^{(i)}(v;x_{pn}) = 0, \quad p=1,2,\dots,n, \ i=1,2,\dots,v-1.$$

Using them, we can write as

$$L_n(v,f;x) = \sum_{k=1}^{n} f(x_{kn})h_{kn}(v;x).$$

Furthermore, we extend the operator $L_n(v,f;x)$. Let l be a nonnegative integer, and let $v-1 \geqslant l$. For $f \in C^{(l)}(\mathbf{R})$ we define the (l,v)-order Hermite–Fejér interpolation polynomials $L_n(l,v,f;x) \in \prod_{vn-1}$ as follows. For each $k=1,2,\dots,n$,

$$L_n(l,v,f;x_{kn}) = f(x_{kn}), \quad L_n^{(j)}(l,v,f;x_{kn}) = f^{(j)}(x_{kn}), \quad j=1,2,\dots,l,$$
$$L_n^{(j)}(l,v,f;x_{kn}) = 0, \quad j=l+1,l+2,\dots,v-1.$$

Especially, $L_n(0,v,f;x)$ is equal to $L_n(v,f;x)$, and for every polynomial $P(x) \in \prod_{vn-1}$ we see $L_n(v-1,v,P;x) = P(x)$. The fundamental polynomials $h_{skn}(v;x) \in \prod_{vn-1}$, $k=1,2,\dots,n$, of $L_n(l,v,f;x)$ are defined by

$$h_{skn}(l,v;x) = l_{kn}^{v}(x) \sum_{i=s}^{v-1} e_{si}(v,k,n)(x-x_{kn})^i, \quad s=0,1,\dots,v-1,$$
$$e_{si} \ (i \leqslant s \leqslant v-1): \text{ real coefficients},$$

$$h_{skn}^{(j)}(l,v;x_{pn}) = \delta_{sj}\delta_{kp}, \quad s=0,1,\dots,v-1, \ p=1,2,\dots,n, \ j=0,1,\dots,v-1. \tag{0.5}$$

Then we have

$$L_n(l,v,f;x) = \sum_{k=1}^{n} \sum_{s=0}^{l} f^{(s)}(x_{kn})h_{skn}(l,v;x).$$

We need some definitions. The Mhaskar–Rahmanov–Saff number a_u is the unique positive root of the equation

$$u = (2/\pi)\int_0^1 a_u t Q'(a_u t)(1-t^2)^{-1/2}\,dt, \quad u>0.$$

We also consider the root $x = q_u > 0$ of $u = xQ'(x)$ for $u > 0$. Let us denote the leading coefficient of the orthonormal polynomial $P_n(W_{rQ}^2; x)$ by γ_n, and then we set $b_n = \gamma_{n-1}/\gamma_n$. Then we have

$$a_n \sim q_n \sim b_n \sim x_{1n}, \quad n = 1, 2, 3, \ldots, \text{ [LL4, Ba, Theorem 3.5]},$$

where if for two sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ there are positive numbers C, D such that $C \leqslant c_n/d_n \leqslant D$, then we denote this fact as $c_n \sim d_n$. We will use the same constant C even if it is different in the same line.

Remark. In previous paper [KaS1] we assumed $r > -1/2$ in (0.3). In this paper we need to suppose $r \geqslant 0$.

1. Estimate of $||P_n(W_{rQ}^2)W_{rQ}||_{L_p(\mathbf{R})}$

In this section we suppose condition (0.1) and $r \geqslant 0$ in (0.3).

Theorem 1.1. *Given $0 < p \leqslant \infty$, we have, for $n \geqslant 1$,*

$$||P_n(W_{rQ}^2)W_{rQ}||_{L_p(\mathbf{R})} \sim a_n^{1/p-1/2} \times \begin{cases} 1, & p < 4, \\ \{\log(1+n)\}^{1/4}, & p = 4, \\ (n^{-2/3})^{1/p-1/4}, & p > 4. \end{cases}$$

When $r = 0$, the result has been obtained by Lubinsky and Moricz [LM]. We may show the following.

Proposition 1.2. *Given $0 < p \leqslant \infty$, we have, for $n \geqslant 1$,*

$$||P_n(W_{rQ}^2)W_{rQn}||_{L_p(\mathbf{R})} \sim a_n^{1/p-1/2} \times \begin{cases} 1, & p < 4, \\ \{\log(1+n)\}^{1/4}, & p = 4, \\ (n^{-2/3})^{1/p-1/4}, & p > 4, \end{cases}$$

where $W_{rQn}(x)$ is defined as follows:

$$W_{rQn}(x) = \begin{cases} (a_n/n)^r W_Q(x) \sim (a_n/n)^r, & |x| < a_n/n, \\ W_{rQ}(x), & a_n/n \leqslant |x|, \end{cases}$$
$$W'_{rQn}(a_n/n) = \lim_{x \to (a_n/n)+0} W'_{rQ}(x). \tag{1.1}$$

In fact, by Kasuga and Sakai [KaS1, Theorem 1.8], we see

$$||P_n(W_{rQ}^2)W_{rQn}||_{L_p(|x| \leqslant a_n/n)} \leqslant o(a_n^{1/p-1/2}).$$

To prove the theorem we repeat the method of [LM], that is, we only check each lemma of [LM], then the theorem is shown easily. First we collect some lemmas, which are shown in previous paper [KaS1]. From now, for simplicity we write $P_n(x) = P_n(W_{rQ}^2; x)$.

Lemma 1.3. *We have the followings*:

(a) *For* $n \geqslant 1$ *and* $x \in \mathbf{R}$,

$$|P_n(x)W_{rQn}(x)| \leqslant Ca_n^{-1/2}/[|1-|x|/a_n|^{1/4} + n^{-1/6}]$$

(*by Kasuga and Sakai* [KaS1, *Theorems* 1.13, 1.14 *and Lemma* 2.7]).

(b) *Let* $0 < p \leqslant \infty$. *There exists* $C > 0$ *such that for* $n \geqslant 1$ *and* $P \in \Pi_n$,

$$||PW_{rQ}||_{L_p(\mathbf{R})} \leqslant C||PW_{rQ}||_{L_p[-a_n, a_n]}$$

(*by Kasuga and Sakai* [KaS1, *Theorem* 1.1]).

(c) *Let* $|x_{jn}| \leqslant \eta a_n$, $0 < \eta < 1$. *There exists a constant* $\delta > 0$ *such that for* $|x - x_{jn}| \leqslant \delta a_n/n$,

$$|P_n'(x)W_{rQn}(x)| \sim na_n^{-3/2}$$ (*by Kasuga and Sakai* [KaS1, *Corollary* 1.12]).

Proposition 1.4. *Let* $0 < p \leqslant \infty$. *There exists* $C > 0$ *such that for* $n \geqslant 2$

$$||P_n(W_{rQ}^2)W_{rQn}||_{L_p(\mathbf{R})} \leqslant Ca_n^{1/p-1/2} \times \begin{cases} 1, & p < 4, \\ \{\log(1+n)\}^{1/4}, & p = 4, \\ (n^{-2/3})^{1/p-1/4}, & p > 4. \end{cases}$$

Proof. It follows from Lemma 1.3(a) and (b) by considering the parts of $|x| \leqslant a_n(1 - n^{-2/3})$ and $a_n(1 - n^{-2/3}) < |x| \leqslant a_n$. □

We need to give the lower bounds.

Lemma 1.5. (a) *For* $n \geqslant 1$,

$$|x_{1n}/a_n - 1| \leqslant Cn^{-2/3}$$ (*by Kasuga and Sakai* [KaS1, *Theorem* 1.3]),

and uniformly for $n \geqslant 3$ *and* $2 \leqslant j \leqslant n-1$

$$x_{j-1,n} - x_{j+1,n} \sim (a_n/n)[\max\{n^{-2/3}, 1 - |x_{j,n}|/a_n\}]^{-1/2}$$

(*by Kasuga and Sakai* [KaS1, *Theorem* 1.4]).

(b) *Uniformly for* $n \geqslant 2$, *and* $1 \leqslant j \leqslant n-1$,

$$\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\} \sim \max\{n^{-2/3}, 1 - |x_{j+1,n}|/a_n\}$$

(*by Kasuga and Sakai* [KaS1, (2.11)]).

(c) *For* $n \geqslant 1$, $1 \leqslant k \leqslant n$ *and* $x \in \mathbf{R}$,

$$|P_n(x)W_{rQ}(x)| \leqslant C(na_n^{-3/2})[\max\{n^{-2/3}, 1-|x|/a_n\}]^{1/4}|x-x_{kn}|$$

(*by Kasuga and Sakai* [KaS1, (2.16)]).

(d) *We have*

$$|P_n(x)W_{rQ}(x)| \leqslant Ca_n^{-1/2}[\max\{n^{-2/3}, 1-|x|/a_n\}]^{-1/4}$$

(*by Kasuga and Sakai* [KaS1, *Theorem* 1.8]).

(e) *Uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n$

$$|P_n'(x_{jn})W_{rQn}(x_{jn})| = |\{P_n(x)W_{rQn}(x)\}'_{x=x_{jn}}| \sim na_n^{-3/2}[\max\{n^{-2/3}, 1-|x_{jn}|/a_n\}]^{1/4}$$

(*by Kasuga and Sakai* [KaS1, (1.8)]).

(f) *Uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n-1$ *and* $x \in \mathrm{R}$,

$$|l_{jn}(x)| \sim (a_n^{3/2}/n)W_{rQn}(x_{jn})[\max\{n^{-2/3}, 1-|x_{jn}|/a_n\}]^{-1/4} \times |P_n(x)/(x-x_{jn})| \quad (\textit{by } (e)).$$

(g) *Uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n-1$ *and* $x \in \mathrm{R}$,

$$|l_{jn}(x)W_{rQn}^{-1}(x_{jn})W_{rQn}(x)| \leqslant C. \tag{1.2}$$

(h) *We have*

$$\max_{|x| \leqslant x_{[n/2],n}} |P_n'(x)| \sim (n/a_n)^r na_n^{-3/2} \quad (\textit{by Kasuga and Sakai } [\text{KaS1, (1.11)}]).$$

Proof. We may only prove (g). First, by Kasuga and Sakai [KaS1, Lemma 2.7] we have

$$||PW_{rQ}||_{L_\infty(|x| \leqslant \delta a_n/n)} \leqslant C||PW_{rQ}||_{L_\infty(\delta a_n/n \leqslant |x| \leqslant a_n)}$$

for $P \in \prod_n$, where $\delta > 0$ is small enough. Therefore, in (c) we can exchange $W_{rQ}(x)$ for $W_{rQn}(x)$. Then, by (f) and (c) we have

$$\begin{aligned} &|l_{jn}(x)W_{rQn}^{-1}(x_{jn})W_{rQn}(x)| \\ &\quad \leqslant (a_n^{3/2}/n)[\max\{n^{-2/3}, 1-|x_{jn}|/a_n\}]^{-1/4} \\ &\qquad \times |P_n(x)W_{rQn}(x)/(x-x_{jn})| \\ &\quad \leqslant C[(\max\{n^{-2/3}, 1-|x|/a_n\})/(\max\{n^{-2/3}, 1-|x_{jn}|/a_n\})]^{1/4}. \end{aligned}$$

If for some fixed $C > 0$,

$$\max\{n^{-2/3}, 1-|x|/a_n\} \leqslant C\max\{n^{-2/3}, 1-|x_{jn}|/a_n\}, \tag{1.3}$$

then we obtain (g). If we set

$$x_{1-s,n} = x_{1n} + sa_n n^{-2/3}; \quad x_{n+s,n} = x_{nn} - sa_n n^{-2/3}, \quad s = 1, 2,$$

then (b) implies (1.3) for $x \in (x_{j-2,n}, x_{j+2,n})$, with a large C. On the other hand, if (1.3) is not true, so that $x \notin (x_{j-2,n}, x_{j+2,n})$, then Lemma 1.3(a) and (e) of this lemma show that

$$\begin{aligned}
&|l_{jn}(x) W_{rQn}^{-1}(x_{jn}) W_{rQn}(x)| \\
&\quad = \left| \left(\frac{P_n(x) W_{rQn}(x)}{x - x_{jn}} \right) \left(\frac{1}{P_n'(x_{jn}) W_{rQn}(x_{jn})} \right) \right| \\
&\quad \leqslant C \left(\frac{a_n^{3/2}}{n} \right) [\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{-1/4} \\
&\qquad \times a_n^{-1/2} [|1 - |x|/a_n|^{1/4} + n^{-1/6}]^{-1} |x_{j\pm 2,n} - x_{jn}|^{-1} \\
&\quad \leqslant C[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{1/4} [|1 - |x|/a_n|^{1/4} + n^{-1/6}]^{-1} \\
&\qquad \text{(by (a) and (b))} \\
&\quad \leqslant C[(\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\})/(\max\{n^{-2/3}, 1 - |x|/a_n\})]^{1/4} \\
&\quad \leqslant C
\end{aligned}$$

for C large enough, as (1.3) does not hold. So we still have (1.2). Hence (g) is true. □

Lemma 1.6 (Cf. Lubinsky and Moricz [LM, p. 49]). *Let* $\eta a_n \leqslant |x_{jn}|$, $0 < \eta < 1$, $x_{n+1,n} = x_{nn}(1 - n^{-2/3})$, $x_{0n} = x_{1n}(1 + n^{-2/3})$. *Then, for* $x \in (x_{j+1,n}, x_{j-1,n})$,

$$\begin{aligned}
&|[(P_n W_{rQ})(x)/\{(x - x_{jn}) P_n'(x_{jn}) W_{rQ}(x_{jn})\}]'| \\
&\quad \leqslant C(n/a_n)[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{1/2}.
\end{aligned}$$

Proof. Let $[z]$ denote the maximum integer nonexceeding z. We may assume $x > 0$. Then we see

$$\begin{aligned}
&[(P_n W_{rQ})(x)/\{(x - x_{jn}) P_n'(x_{jn}) W_{rQ}(x_{jn})\}]' \\
&\quad = [x^{r-[r+1]} (x^{[r+1]} P_n W_Q)(x)/\{(x - x_{jn}) P_n'(x_{jn}) W_{rQ}(x_{jn})\}]' \\
&\quad = (r - [r+1]) x^{r-[r+1]-1} x^{[r+1]} (P_n W_Q)(x)/\{(x - x_{jn}) P_n'(x_{jn}) W_{rQ}(x_{jn})\} \\
&\qquad + x^{r-[r+1]} [x^{[r+1]} (P_n W_Q)(x)/\{(x - x_{jn}) P_n'(x_{jn}) W_{rQ}(x_{jn})\}]'.
\end{aligned}$$

Here, by Lemma 1.5(g),

$$\begin{aligned}
&|(r - [r+1]) x^{r-[r+1]-1} x^{[r+1]} (P_n W_Q)(x)/\{(x - x_{jn}) P_n'(x_{jn}) W_{rQ}(x_{jn})\}| \\
&\quad \leqslant C a_n^{-1} |(P_n W_{rQ})(x)/\{(x - x_{jn}) P_n'(x_{jn}) W_{rQ}(x_{jn})\}| \leqslant C a_n^{-1}. \qquad (1.4)
\end{aligned}$$

Furthermore, by the Markov–Bernstein inequality [LM, Lemma 2.4],

$$\begin{aligned}
&|x^{r-[r+1]}[x^{[r+1]}(P_n W_Q)(x)/\{(x-x_{jn})P_n'(x_{jn})W_{rQ}(x_{jn})\}]'| \\
&\leqslant C(n/a_n)|x^{r-[r+1]}|\left\|\left\{\frac{t^{[r+1]}(P_n W_Q)(t)}{(t-x_{jn})P_n'(x_{jn})W_{rQ}(x_{jn})}\right\}\right\|_{L_\infty(\mathbf{R})} \\
&\quad\times[\max\{n^{-2/3},1-|x|/a_n\}]^{1/2}. \qquad (1.5)
\end{aligned}$$

For $0<t\leqslant 2a_n$,

$$\begin{aligned}
&|x^{r-[r+1]}t^{[r+1]}(P_n W_Q)(t)/\{(t-x_{jn})P_n'(x_{jn})W_{rQ}(x_{jn})\}| \\
&=|(t/x)^{[r+1]-r}||(P_n W_{rQ})(t)/\{(t-t_{jn})P_n'(x_{jn})W_{rQ}(x_{jn})\}| \\
&\leqslant C \quad \text{(by Lemma 1.5(g))}.
\end{aligned}$$

For $2a_n<t$,

$$\begin{aligned}
&|x^{r-[r+1]}||t^{[r+1]}(P_n W_Q)(t)/\{(t-x_{jn})P_n'(x_{jn})W_{rQ}(x_{jn})\}| \\
&\leqslant C|x^{r-[r+1]}||(P_n W_{rQ})(t)/\{t^{1-[r+1]+r}P_n'(x_{jn})W_{rQ}(x_{jn})\}| \\
&\leqslant Ca_n^{-1/2}n^{1/6}/(a_n n a_n^{-3/2}n^{-1/6}) \\
&\leqslant Cn^{-2/3} \quad \text{(by Lemma 1.5(d) and (e))}.
\end{aligned}$$

Therefore, by (1.5) we have

$$\begin{aligned}
&|x^{r-[r+1]}[x^{[r+1]}(P_n W_Q)(x)/\{(x-x_{jn})P_n'(x_{jn})W_{rQ}(x_{jn})\}]'| \\
&\leqslant C(n/a_n)[\max\{n^{-2/3},1-|x|/a_n\}]^{1/2}.
\end{aligned}$$

Here, since $x\in(x_{j+1,n},x_{j-1,n})$ we see by Lemma 1.5(b),

$$[\max\{n^{-2/3},1-|x|/a_n\}]^{1/2}\sim[\max\{n^{-2/3},1-|x_j|/a_n\}]^{1/2},$$

consequently, with (1.4) we have the lemma. □

Lemma 1.7 (Kasuga amd Sakai [KaS1, Corollary 1.12]). *Let* $|x_{in}|\leqslant\eta a_n$, $0<\eta<1$.

(i) *Let n be odd. For* $\delta a_n/n\leqslant|x|\leqslant x_{[n/2],n}$, $\delta>0$,

$$|P_n(x)|\sim(n/a_n)^r\ a_n^{-1/2},$$

and there is a constant $\delta'>0$ *such that for* $|x|\leqslant\delta' a_n/n$,

$$|P_n'(x)|\sim(n/a_n)^r n a_n^{-3/2}.$$

Let n be even. For $-x_{[n/2],n}+\delta a_n/n\leqslant x\leqslant x_{[n/2],n}-\delta a_n/n$, $\delta>0$, *we see*

$$|P_n(x)W_{rQ}(x)|\sim a_n^{-1/2}.$$

(ii) *Let* $x_{kn}\geqslant 0$ *or* $x_{k-1,n}\leqslant 0$. *For* $x_{kn}+\delta a_n/n\leqslant x\leqslant x_{k-1,n}-\delta a_n/n$, $\delta>0$, *we see*

$$|P_n(x)W_{rQ}(x)|\sim a_n^{-1/2},$$

and there is a constant $\delta' > 0$ *such that for* $x_{kn} - \delta' a_n/n \leqslant |x| \leqslant x_{kn} + \delta' a_n/n$,

$$|P_n'(x) W_{rQ}(x)| \sim n a_n^{-3/2}.$$

Lemma 1.8. *There exists* $C > 0$ *such that uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n$, *and for*

$$|x - x_{jn}| \leqslant C(a_n/n)[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{-1/2},$$

we have

$$|P_n(x) W_{rQn}(x)| \sim (n a_n^{-3/2})[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{1/4} |x - x_{jn}|.$$

Proof. If $x_{jn} = 0$ (that is n is odd), then we have the lemma by using Lemmas 1.5(h) and 1.7(i). Therefore, we may assume $x_{jn} \neq 0$. We consider the polynomial

$$\tau_{jn}(x) = l_{jn}(x) W_{rQn}^{-1}(x_{jn}).$$

We have $(\tau_{jn} W_{rQn})(x_{jn}) = 1$, and by Lemma 1.5(g) we see $||\tau_{jn} W_{rQn}||_{L_\infty(\mathrm{R})} \leqslant C$, with C independent of j and n. Here let $\eta > 0$ be fixed, and let

$$\varepsilon_n = \varepsilon(j, n) = \eta(a_n/n)[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{-1/2}. \tag{1.6}$$

We use $x_{1-s,n}$ and $x_{n+s,n}$, $s = 1, 2$, which are defined in the proof of Lemma 1.5(e). Now if η is small enough, Lemma 1.5(a) shows that uniformly for $1 \leqslant j \leqslant n$

$$(x_{jn} - \varepsilon_n,\ x_{jn} + \varepsilon_n) \subset (x_{j+2,n} + \varepsilon_n,\ x_{j-2,n} - \varepsilon_n). \tag{1.7}$$

Let $\eta a_n < |x_{jn}|$, $0 < \eta < 1$. Then for $x \in (x_{jn} - \varepsilon_n, x_{jn} + \varepsilon_n)$, Lemma 1.6 shows that

$$|(\tau_{jn} W_{rQn})'(x)| \leqslant C(n/a_n)[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{1/2}.$$

If $t \in (x_{jn} - \varepsilon_n,\ x_{jn} + \varepsilon_n)$, we have, for some ξ between t and x_{jn},

$$\begin{aligned} |(\tau_{jn} W_{rQn})(t)| &= |(\tau_{jn} W_{rQn})(x_{jn}) + (\tau_{jn} W_{rQn})'(\xi)(t - x_{jn})| \\ &\geqslant 1 - C(n/a_n)[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{1/2} \varepsilon_n \\ &= 1 - C\eta \geqslant 1/2 \end{aligned}$$

when η of (1.6) is small enough. Therefore,

$$|(\tau_{jn} W_{rQn})(t)| \sim 1, \quad t \in (x_{jn} - \varepsilon_n,\ x_{jn} + \varepsilon_n), \tag{1.8}$$

and by Lemma 1.5(f) and the definition of $\tau_{jn}(x)$ we have the lemma.

Let $|x_{jn}| \leqslant \eta a_n$, $0 < \eta < 1$. Then by Lemma 1.3(c) we have (1.8). In fact, by Lemma 1.7, for $t \in (x_{jn} - \varepsilon_n,\ x_{jn} + \varepsilon_n)$,

$$\begin{aligned} |(\tau_{jn} W_{rQn})(t)| &= |(P_n W_{rQn})(t)/\{(t - x_{jn}) P_n'(x_{jn}) W_{rQn}(x_{jn})\}| \\ &= |(P_n W_{rQn})'(\xi)/\{P_n'(x_{jn}) W_{rQn}(x_{jn})\}| \\ &\quad \times (|\xi - x_{jn}| < |t - x_{jn}| < \delta a_n/n) \\ &\geqslant C > 0 \end{aligned}$$

(by $|(P_n W_{rQn})'(\xi)| \geqslant (1/2)|(P_n' W_{rQn})(\xi)|$ for δ small enough). Therefore, we also obtain (1.8), and so by Lemma 1.5(f) and the definition of $\tau_{jn}(x)$ we have the lemma. $\square$

Remark 1.9. By (1.8), we have, for $j = 2, 3, \ldots, n$,

$$x_{j-1,n} - x_{j,n} \sim (a_n/n)[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{-1/2}. \tag{1.9}$$

In fact, we see $(\tau_{jn} W_{rQn})(x_{j-1,n}) = 0$. If $x_{j-1,n} \in (x_{jn} - \varepsilon_n,\ x_{jn} + \varepsilon_n)$, then by (1.8) we see $(\tau_{jn} W_{rQn})(x_{j-1,n}) \neq 0$. But this contradicts. Therefore, we have $x_{j-1,n} \notin (x_{jn} - \varepsilon_n,\ x_{jn} + \varepsilon_n)$. From this and Lemma 1.5(a) we have (1.9).

Proof of Proposition 1.2. We fix j as $1 \leqslant j \leqslant n$. Let C be the constant in Lemma 1.8, and let us consider ε_n with $\eta = C$ in (1.6). First let $x_{j+2,n} > 0$ or $x_{j-2,n} < 0$. By (1.7) and Lemma 1.8 we have

$$\begin{aligned}
&\int_{x_{j+2,n}}^{x_{j-2,n}} |(P_n W_{rQn})(x)|^p\, dx \\
&\geqslant C \int_{x_{jn}-\varepsilon_n}^{x_{jn}+\varepsilon_n} [(n a_n^{-3/2})(\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\})^{1/4} |x - x_{jn}|]^p\, dx \\
&\geqslant C[(n a_n^{-3/2})(\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\})^{1/4}]^p \varepsilon_n^{p+1} \\
&\geqslant C(a_n^{1-p/2}/n)[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{-p/4-1/2} \\
&\geqslant C a_n^{-p/2}(x_{j-2,n} - x_{j+2,n})[\max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}]^{-p/4} \quad \text{(by Lemma 1.5(a))} \\
&\geqslant C a_n^{-p/2} \int_{x_{j+2,n}}^{x_{j-2,n}} [\max\{n^{-2/3}, 1 - |t|/a_n\}]^{-p/4}\, dt
\end{aligned}$$

in view of Lemma 1.5(b). Let $x_{jn} = 0$. Then by definition (1.1) and Lemma 6 we see

$$\begin{aligned}
&\int_{x_{j+2,n}}^{x_{j-2,n}} |(P_n W_{rQn})(x)|^p\, dx \\
&\geqslant C \int_{x_{[n/2],n}-\varepsilon a_n/n}^{x_{[n/2],n}+\varepsilon a_n/n} \{a_n^{-1/2}\}^p\, dx \quad \text{(fixed } \varepsilon > 0 \text{ small enough)} \\
&\geqslant C a_n^{-p/2} \int_{x_{j+2,n}}^{x_{j-2,n}} [\max\{n^{-2/3}, 1 - |t|/a_n\}]^{-p/4}\, dt.
\end{aligned}$$

In the case of $x_{in} = 0$, $i = j-1$ or $j+1$ we also have the same estimate described above. Summing, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} |(P_n W_{rQn})(x)|^p\, dx \\
&\geqslant C a_n^{-p/2} \int_{x_{nn}}^{x_{1n}} [\max\{n^{-2/3}, 1 - |t|/a_n\}]^{-p/4}\, dt
\end{aligned}$$

$$= Ca_n^{1-p/2} \int_{x_{nn}/a_n}^{x_{1n}/a_n} [\max\{n^{-2/3}, 1-|s|\}]^{-p/4}\, ds$$

$$\geqslant Ca_n^{1-p/2} \int_{-1+Cn^{-2/3}}^{1-Cn^{-2/3}} (1-|s|)^{-p/4}\, ds \quad \text{(by Lemma 1.5(a))}$$

$$\geqslant Ca_n^{1-p/2} \times \begin{cases} 1, & p<4, \\ \log(1+n), & p=4, \\ (n^{-2/3})^{1-p/4}, & p>4. \end{cases}$$

Hence,

$$||P_n(W_{rQ}^2)W_{rQn}||_{L_p(\mathrm{R})} \geqslant Ca_n^{1/p-1/2} \times \begin{cases} 1, & p<4, \\ \{\log(1+n)\}^{1/4}, & p=4, \\ (n^{-2/3})^{1/p-1/4}, & p>4. \end{cases}$$

Therefore, from Proposition 1.4 we have Proposition 1.2. □

Theorem 1.1 is shown by Proposition 1.2.

2. Markov inequalities

In this section we show the Markov inequalities, which are used in the next section. In this section we suppose $r \geqslant 0$. For the Freud weight $W_Q(x) = \exp(-Q(x))$ we know the following theorems.

Theorem A (Levin and Lubinsky [LL5, Remarks (a) of Theorem 1.1]). *Let Q satisfy* (0.1) *for A, $B>1$, and let $1 \leqslant p < \infty$. Then there exists a constant $C>0$ such that for $P \in \prod_n$,*

$$||P'W_Q||_{L_p(\mathbf{R})} \leqslant C(n/a_n)||PW_Q||_{L_p(\mathbf{R})}.$$

Theorem B (Levin and Lubinsky [LL3, Theorem 1.1]). *Let Q satisfy* (0.1) *for A, $B>0$. Then there exists a constant $C>0$ such that for $P \in \prod_n$,*

$$||P'W_Q||_{L_\infty(\mathbf{R})} \leqslant \left\{ \int_1^{Cn} (1/Q^{[-1]}(s))\, ds \right\} ||PW_Q||_{L_\infty(\mathbf{R})},$$

where $Q^{[-1]}(x)$ denotes the inverse function of $Q(x)$.

Especially if $1 < A \leqslant B$, then we have

$$||P'W_{rQ}||_{L_\infty(\mathbf{R})} \leqslant C(n/a_n)||PW_{rQ}||_{L_\infty(\mathbf{R})}.$$

In fact, we see

$$\int_1^{Cn} (1/Q^{[-1]}(s))\, ds \sim n/a_n \quad \text{[LL4, } \textit{Lemma } \text{5.2(f)].}$$

We obtain analogies of Theorems A and B for the weight $W_{rQ}(x)$ $(x\in\mathbf{R}, r\geqslant 0)$, where $Q(x)$ is the Freud exponent satisfying (0.1) for $A, B>1$.

Theorem 2.1. *Let Q satisfy* (0.1) *for $A, B>1$, and let $1\leqslant p\leqslant\infty$. Then there exists a constant $C>0$ such that for $P\in\prod_n$,*

$$||P'W_{rQ}||_{L_p(\mathbf{R})}\leqslant C(n/a_n)||PW_{rQ}||_{L_p(\mathbf{R})}.$$

To show the theorem we use the idea of Freud and Levin–Lubinsky [LL1,LL2]. We need some simple lemmas. Let $0\leqslant\delta<2$, and let $1<\lambda$ be large enough. For $0<\delta<2$ we define a continuously differentiable function

$$\phi_n(\delta,\lambda;t)=\begin{cases}|t|^\delta & (\lambda/n\leqslant|t|\leqslant 1),\\ (\delta/2)(\lambda/n)^{\delta-2}t^2+(1-\delta/2)(\lambda/n)^\delta & (|t|\leqslant\lambda/n),\end{cases}$$

and we set $\phi_n(0,\lambda;t)=1$. From now, we may assume $0<\delta<2$.

Lemma 2.2. *For λ large enough there exist a polynomial $T_n(\delta,\lambda;t)\in\prod_n$, and constants $C_1(\lambda), C_2(\lambda), C_3(\lambda)>0$ such that*

$$C_1(\lambda)\leqslant|T_n(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)|\leqslant C_2(\lambda),$$
$$|T_n'(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)|\leqslant C_3(\lambda)n.$$

Proof. By Jackson's theorem [Ja] we see that there exist $T_n(\delta,\lambda;t)\in\prod_n$ and a constant C independent of ϕ_n such that

$$|T_n(\delta,\lambda;t)-\phi_n(\delta,\lambda;t)|\leqslant C(1/n)\omega(\phi_n'(\delta,\lambda);1/n),$$
$$|T_n'(\delta,\lambda;t)-\phi_n'(\delta,\lambda;t)|\leqslant C\omega(\phi_n'(\delta,\lambda);1/n),$$

where $\omega(f,h)$ is the modulus of continuity for f. Here, we see

$$|\phi_n'(\delta,\lambda;t+1/n)-\phi_n'(\delta,\lambda;t)|\leqslant C\lambda^{\delta-2}(1/n)^{\delta-1}.$$

Therefore, we see

$$\begin{aligned}|T_n(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)-1|&\leqslant C(1/\phi_n(\delta,\lambda;t))(1/n)\lambda^{\delta-2}(1/n)^{\delta-1}\\ &\leqslant C[2/\{(2-\delta)\lambda^2\}]\leqslant 1/2\end{aligned}$$

for λ large enough. Hence, we have

$$C_1(\lambda)\leqslant|T_n(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)|\leqslant C_2(\lambda).$$

Similarly, for λ large enough we have

$$\begin{aligned}&|T_n'(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)-\phi_n'(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)|\\ &\quad\leqslant C(1/\phi_n(\delta,\lambda;t))\omega(\phi_n'(\delta,\lambda);1/n)\\ &\quad\leqslant(1/2)n.\end{aligned}$$

Here, we see

$$|\phi_n'(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)| \leqslant \{2/(2-\delta)\}(\delta n/\lambda),$$

therefore, we have, for λ large enough,

$$|T_n'(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)| \leqslant n/2 + \{(2\delta)/(2-\delta)\}(n/\lambda) \leqslant C_3(\lambda)n. \qquad \square$$

We set $x = 2a_n t$. We define a differentiable function

$$\begin{aligned} \Phi_n(\delta,\lambda;x) &= (2a_n)^{\delta}\phi_n(\delta,\lambda;t) \\ &= \begin{cases} |x|^{\delta} \quad (2\lambda a_n/n \leqslant |x| \leqslant 2a_n), \\ (2a_n)^{\delta}[(\delta/2)(2\lambda/n)^{\delta-2}\{x^2/(2a_n)^2\} + (1-\delta/2)(\lambda/n)^{\delta}] \\ \quad (|x| \leqslant 2\lambda a_n/n), \end{cases} \end{aligned} \tag{2.1}$$

and set

$$S_n(\delta,\lambda;x) = (2a_n)^{\delta} T_n(\delta,\lambda;t). \tag{2.2}$$

From Lemma 2.2 we see the following.

Lemma 2.3. *Let $x = 2a_n t$, then for $2\lambda a_n/n \leqslant |x| \leqslant 2a_n$,*

$$\Phi_n(\delta,\lambda;x) \sim (2a_n)^{\delta}\phi_n(\delta,\lambda;t) \sim |x|^{\delta},$$

and

$$S_n(\delta,\lambda;x) \sim (2a_n)^{\delta} T_n(\delta,\lambda;t) \sim |x|^{\delta}.$$

From Lemmas 2.2 and 2.3 we conclude the following.

$$|S_n(\delta,\lambda;x)/\Phi_n(\delta,\lambda;x)| = |T_n(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)|.$$

We have for $|x| \leqslant 2a_n$,

$$C_1(\lambda) \leqslant |S_n(\delta,\lambda;x)/\Phi_n(\delta,\lambda;x)| \leqslant C_2(\lambda). \tag{2.3}$$

Furthermore, we see

$$\begin{aligned} &|S_n'(\delta,\lambda;x)/\Phi_n(\delta,\lambda;x)| \\ &\quad = |(1/2a_n)T_n'(\delta,\lambda;t)/\phi_n(\delta,\lambda;t)| \leqslant C_3(\lambda)\{n/(2a_n)\}. \end{aligned} \tag{2.4}$$

Let $1 \leqslant p \leqslant \infty$, and let the constants in (0.1) satisfy $A, B > 1$. We know $a_{2n} < 2a_n$ (see [LL1,LL2,LL3,LL4,LL5, Proof of Lemma 5.2(c)]). So we use Lemma 2.3 for $2\lambda a_n/n \leqslant |x| \leqslant 2a_n$.

Lemma 2.4 (Kasuga and Sakai [KaS1, Lemma 2.7]). *We assume that $pr+1>0$ if $0<p<\infty$, and $r \geqslant 0$ if $p=\infty$. There exist constants ε, $C>0$ such that for every $P \in \prod_n$ and $n = 0, 1, 2, \ldots$, we have*

$$\|PW_{rQ}\|_{L_p(|x| \leqslant \varepsilon a_n/n)} \leqslant C\|PW_{rQ}\|_{L_p(\varepsilon a_n/n \leqslant |x| \leqslant a_n)},$$

where $0<p \leqslant \infty$.

Now, we prove Theorem 2.1. We use the following modified weights.

For $0<\delta<2$ we define

$$W_{\delta Qn,\lambda}=\begin{cases} W_{\delta Q}(x) & (\lambda a_n/n\leqslant|x|),\\ W_{\delta Q}(\lambda a_n/n) & (|x|\leqslant\lambda a_n/n),\end{cases} \tag{2.5}$$

where $\lambda>0$ is fixed large enough.

Proof of Theorem 2.1. We take $\lambda>0$ large enough, and we consider the function $\Phi_n(\delta,\lambda;x)$ as defined in (2.1), and the polynomial $S_n(\delta,\lambda;x)$ as defined in (2.2). Let $1\leqslant p\leqslant\infty$, $P\in\Pi_n$.

First, let $0\leqslant r=\delta<2$. We use Lemma 2.3. By (2.3), we have for $|x|\leqslant a_{2n}$,

$$\begin{aligned}|P'(x)W_{\delta Q}(x)|&\leqslant C|P'(x)\Phi_n(\delta,\lambda;x)W_Q(x)|\\ &\leqslant C|P'(x)S_n(\delta,\lambda;x)W_Q(x)|\\ &\leqslant C|\{P(x)S_n(\delta,\lambda;x)\}'W_Q(x)-P(x)S_n'(\delta,\lambda;x)W_Q(x)|.\end{aligned}$$

So by Lemma 2.4, with ε small enough, and by the infinite–finite range inequality, we have for $1\leqslant p\leqslant\infty$,

$$\begin{aligned}&||P'(x)W_{\delta Q}(x)||_{L_p(\mathbf{R})}\\ &\quad\leqslant C||P'(x)W_{\delta Q}(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant 2a_n)}\\ &\quad\leqslant C(n/a_n)||\{P(x)S_n(\delta,\lambda;x)\}'W_Q(x)||_{L_p(\mathbf{R})}\\ &\qquad+||P(x)S_n'(\delta,\lambda;x)W_Q(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant 2a_n)}\\ &\quad\leqslant C(n/a_n)||P(x)S_n(\delta,\lambda;x)W_Q(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant a_{2n})}\\ &\qquad+C(n/a_n)||P(x)\Phi_n(\delta,\lambda;x)W_Q(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant 2a_n)}\\ &\qquad\text{(by Lemma 2.4 and (2.4))}\\ &\quad\leqslant C(n/a_n)||P(x)S_n(\delta,\lambda;x)W_Q(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant a_{2n})}\\ &\qquad+C(n/a_n)||P(x)S_n(\delta,\lambda;x)W_Q(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant 2a_n)}\\ &\qquad\text{(by (2.4))}\\ &\quad\leqslant C(n/a_n)||P(x)W_{\delta Qn,2\lambda}(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant a_{2n})}\\ &\qquad\text{(by Lemma 2.3, and see (2.5))}\\ &\quad\leqslant C(n/a_n)||P(x)W_{\delta Q}(x)||_{L_p(\mathbf{R})}.\end{aligned}$$

Here, we used the fact

$$W_{\delta Qn,2\lambda}(x)\sim W_{\delta Q}(x)\quad\text{for }\varepsilon a_n/n\leqslant|x|\leqslant 2\lambda a_n/n.$$

Now, for the general case we set for $0<r$,

$$r=2m+\delta,\quad m=0,1,2,\ldots,\quad 0\leqslant\delta<2.$$

Then, we have by the infinite–finite range inequality and Lemma 2.4,

$$\begin{aligned}|P'(x)W_{rQ}(x)||_{L_p(\mathbf{R})} &\leqslant C||P'(x)x^{2m}W_{\delta Q}(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant 2a_n)}\\ &\leqslant C[||\{P(x)x^{2m}\}'W_{\delta Q}(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant 2a_n)}\\ &\quad + 2m||P(x)x^{2m-1}W_{\delta Q}(x)||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant 2a_n)}]\\ &\leqslant C[(n/a_n)||P(x)x^{2m}W_{\delta Q}(x)||_{L_p(\mathbf{R})}\\ &\quad + (\varepsilon a_n/n)^{-1}||P(x)x^{2m}W_{\delta Q}(x)||_{L_p(\mathbf{R})}]\\ &\leqslant C(n/a_n)||P(x)W_{rQ}(x)||_{L_p(\mathbf{R})},\end{aligned}$$

where $C = C(\varepsilon)$. □

3. Hermite–Fejér interpolation polynomials

Our main purpose in this section is to give estimates of the coefficients $e_i(v,k,n)$, $e_{si}(v,k,n)$, $s = 0,1,\ldots,v-1$, of fundamental polynomial $h_{kn}(v;x)$ or $h_{kn}(l,v;x)$. In the next section we give the proofs of theorems. We supposed $r > -1/2$ in (0.3). The results are important for studies of convergence or divergence of the higher order Hermite–Fejér interpolation polynomials. For the typical case $W_m(x) = \exp(-|x|^m)$, $m = 1,\ldots,$ we have obtained some convergence or divergence theorems in [KS1,KS2]. We can also obtain the same result for $L_n(v,f;x)$ with the weights (0.3). In Section 5 we will report some applications.

We define

$$\langle i\rangle = \begin{cases}1 & (i\text{: odd}),\\ 0 & (i\text{: even}),\end{cases}\qquad M_n(Q;x) = |x|/a_n^2 + |Q'(x)|.$$

To get the estimate of coefficients $e_i(v,k,n)$ we need the following theorem.

Theorem 3.1. *Let Q satisfy the condition $C(v+1)$. For $i = 1,2,\ldots,v-1$ we have*

$$|(l_{kn}^v)^{(i)}(x_{kn})| \leqslant C\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}^{\langle i\rangle}(n/a_n)^{i-\langle i\rangle},\quad x_{kn}\neq 0.$$

For $x_{kn} = 0$ we see

$$|(l_{kn}^v)^{(i)}(0)| \leqslant C(n/a_n)^i.$$

Corollary 3.2. *If Q satisfies the condition $C(v)$, for $i = 1,2,\ldots,v-1$,*

$$|(l_{kn}^v)^{(i)}(x_{kn})| \leqslant C(n/a_n)^i,\quad k = 1,2,\ldots,n.$$

Theorem 3.3. *Let Q satisfy the condition $C(v+1)$. For $i = 1, 2, \ldots, v-1$, we have*

$$e_0(v,k,n) = 1,$$
$$|e_i(v,k,n)| \leqslant C\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}^{\langle i \rangle}(n/a_n)^{i-\langle i \rangle}, \quad x_{kn} \neq 0.$$

For $x_{kn} = 0$ we see $e_0(v,k,n) = 1$, $|e_i(v,k,n)| \leqslant C(n/a_n)^i$, $i = 1, 2, 3, \ldots, v-1$.

Corollary 3.4. *If Q satisfies the condition $C(v)$, for $i = 1, 2, \ldots, v-1$,*

$$e_0(v,k,n) = 1, \quad |e_i(v,k,n)| \leqslant C(n/a_n)^i, \quad i = 1, 2, \ldots, v-1, \ k = 1, 2, \ldots, n.$$

The coefficients $e_{si}(l,v,k,n)$ have the following estimates.

Theorem 3.5. *If Q satisfies the condition $C(v)$, then we have*

$$e_{ss}(l,v,k,n) = 1/s!, \quad |e_{si}(l,v,k,n)| \leqslant C(n/a_n)^{i-s},$$
$$i = s, s+1, \ldots, v-1, \ s = 0, 1, \ldots, v-1, \ k = 1, 2, \ldots, n.$$

The following theorem is important to show a divergence theorem with respect to $L_n(v,f;x)$.

Theorem 3.6 (Cf. Kanjin and Sakai [KS1, (4.16)], Sakai and Vértesi [SV]). *Let Q satisfy the condition $C(v+1)$, and let $v \geqslant 1$ be odd. For $j = 0, 1, 2, \ldots$, there is a polynomial $\Psi_j(x)$ of degree j such that $(-1)^j \Psi_j(-\mu) > 0$ for $\mu = 1, 3, 5, \ldots$, and the following relation holds. Let $0 < \varepsilon$ (small enough). Then we have an expression*

$$e_{2s}(v,k,n) = (-1)^s\{1/(2s)!\}\Psi_s(-v)\beta_n^s(k)(n/a_n)^{2s} \times \{1 + \eta_{kn}(v,s)\}, \quad s = 0, 1, \ldots, (v-1)/2. \tag{3.1}$$

Here $0 < D_1 \leqslant \beta_n(k) \leqslant D_2$ (D_1 and D_2 are independent of n and k), and $\eta_{kn}(v,s)$ satisfies

$$|\eta_{kn}(v,s)| \leqslant C \max(\varepsilon, \varepsilon^{A-1}) \tag{3.2}$$

for k with $(1/\varepsilon)(a_n/n) \leqslant |x_{kn}| \leqslant \varepsilon a_n$, where A is a constant defined in (0.1), and the constant C is independent of n, k and ε.

4. Proofs of theorems

In this section we prove the results in Section 3. We use some results in [KaS1].

Lemma 4.1 (Kasuga and Sakai [KaS1, Theroem 3.6]). *If Q satisfies the condition $C(v+1)$, then for $i = 1, 2, \ldots, v$ and $x_{kn} \neq 0$ we have*

$$|P_n^{(i)}(x_{kn})| \leqslant C\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}^{1-\langle i \rangle}(n/a_n)^{i-2+\langle i \rangle}|P_n'(x_{kn})|.$$

If $x_{kn}=0$ (*that is*, *n odd*), *then*

$$|P_n^{(i)}(0)|\leqslant C(n/a_n)^{i-1}|P_n'(0)|,\quad i=1,2,\ldots,v.$$

Proof of Theorem 3.1. We use an induction with respect to v. Let $x_{kn}\neq 0$. Obviously

$$\begin{aligned}l_{kn}(x)&=P_n(x)/\{(x-x_{kn})P_n'(x_{kn})\}\\&=\{1/P_n'(x_{kn})\}[\{P_n'(x_{kn})/1!\}+\{P_n''(x_{kn})(x-x_{kn})/2!\}+\cdots\\&\quad+\{P_n^{(n)}(x_{kn})(x-x_{kn})^{n-1}/n!\}].\end{aligned}$$

From Lemma 4.1

$$\begin{aligned}|\{l_{kn}(x)\}^{(i)}_{x=x_{kn}}|&=|P_n^{(i+1)}(x_{kn})/\{(i+1)P_n'(x_{kn})\}|\\&\leqslant C\{M_n(Q;x_{kn})+1/|x_{kn}|\}^{1-\langle i+1\rangle}(n/a_n)^{i-\langle i\rangle}\\&\leqslant C\{M_n(Q;x_{kn})+1/|x_{kn}|\}^{\langle i\rangle}(n/a_n)^{i-\langle i\rangle}.\end{aligned}$$

We assume that the theorem is true for a certain $v\geqslant 1$. Then

$$\begin{aligned}|\{l_{kn}^v(x)\}^{(i)}_{x=x_{kn}}|&=\left|\sum_{s=0}^{i}\binom{i}{s}(l_{kn}^{v-1})^{(s)}(x_{kn})(l_{kn})^{(i-s)}(x_{kn})\right|\\&\leqslant C\sum_{s=0}^{i}\{M_n(Q;x_{kn})+1/|x_{kn}|\}^{\langle s\rangle+\langle i-s\rangle}(n/a_n)^{i-\langle s\rangle-\langle i-s\rangle}\\&\leqslant C\{M_n(Q;x_{kn})+1/|x_{kn}|\}^{\langle i\rangle}(n/a_n)^{i-\langle i\rangle}.\end{aligned}$$

For $x_{kn}=0$ we can show the theorem similarly. □

Proof of Corollary 3.2. This is trivial by Theorem 3.1, because $M_n(Q;x_{kn})+1/|x_{kn}|\leqslant Cn/a_n$. □

Here we can estimate the coefficients $e_i(v,k,n)$ of the fundamental polynomials $h_{kn}(v;x)$.

Proof of Theorem 3.3. *Let* $x_{kn}\neq 0$. *Obviously* $e_0(v,k,n)=1$. *Using the properties of* $h_{kn}(v;x)$, *for* $i>0$,

$$\begin{aligned}e_i(v,k,n)&\leqslant C\sum_{s=0}^{i-1}|e_s(v,k,n)||(l_{kn}^v)^{(i-s)}(x_{kn})|\\&\leqslant C\sum_{s=0}^{i-1}\{M_n(Q;x_{kn})+1/|x_{kn}|\}^{\langle s\rangle}(n/a_n)^{s-\langle s\rangle}\end{aligned}$$

$$\times \{M_n(Q;x_{kn}) + 1/|x_{kn}|\}^{\langle i-s\rangle}(n/a_n)^{i-s-\langle i-s\rangle}$$
$$\leqslant C\sum_{s=0}^{i-1}\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}^{\langle s\rangle+\langle i-s\rangle}(n/a_n)^{i-\langle s\rangle-\langle i-s\rangle}$$
$$\leqslant C\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}^{\langle i\rangle}(n/a_n)^{i-\langle i\rangle}.$$

For $x_{kn} = 0$ *we can show the result similarly.* □

Proof of Corollary 3.4. *From Theorem* 2.3 *the corollary is trivial, because* $M_n(Q;x_{kn}) + 1/|x_{kn}| \leqslant Cn/a_n$. □

Using the method of proving Theorem 3.3, we can show Theorem 3.5.

Proof of Theorem 3.5. We prove it by induction for i. From $h_{skn}(l,v,x_{kn}) = 1$, it follows that $e_{ss}(l,v,k,n) = 1/s!$, so the case $i = s$ holds. By (0.5) and the fact $h_{skn}^{(i)}(l,v,x_{kn}) = 0$, $s+1 \leqslant i \leqslant v-1$, we easily see

$$e_{is}(l,v,k,n) = -\sum_{p=s}^{i-1}\{1/(i-p)!\}e_{ps}(l,v,k,n)(l_{kn}^{v})^{(i-p)}(x_{kn}),$$
$$s+1 \leqslant i \leqslant v-1.$$

Since $M_n(x_{kn}) \leqslant C(n/a_n)$, it follows from Corollary 3.2 that $|(l_{kn}^{v})^{(s)}(x_{kn})| \leqslant C(a_n/n)^{-s}$ for every s, where C is independent of n and k. This inequality and the assumption of induction lead to

$$|e_{is}(l,v,k,n)| \leqslant C\sum_{p=s}^{i-1}|e_{ps}(l,v,k,n)||(l_{kn}^{v})^{(i-p)}(x_{kn})|$$
$$\leqslant C\sum_{p=s}^{i-1}(n/a_n)^{p-s}(n/a_n)^{i-p} \leqslant C(n/a_n)^{i-s},$$

where C is independent of n and k. □

Next, we show Theorem 3.6. The method of proving is an analogy of [KS1], therefore we only sketch the proof simply.

We define

$$M_n^*(Q;x) = \begin{cases} |x|/a_n^2 + |Q'(x)| + 1/|x|, & x \neq 0, \\ (n/a_n), & x = 0. \end{cases} \tag{4.1}$$

We need some lemmas.

Lemma 4.2 (Kasuga and Sakai [KaS1, Theorem 1.6]). *We have an expression*

$$P_n'(x) = A_n(x)P_{n-1}(x) - B_n(x)P_n(x) - 2r\{P_n(x)/x\}^*,$$

where

$$
\begin{aligned}
A_n(x) &= 2b_n \int_{-\infty}^{\infty} P_n^2(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt,\\
B_n(x) &= 2b_n \int_{-\infty}^{\infty} P_n(t)P_{n-1}(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt,\\
\{P_n(x)/x\}^* &= \begin{cases} P_n(x)/x & (n\text{: odd}),\\ 0 & (n\text{: even}),\end{cases}\\
\bar{Q}(x,t) &= \{Q'(t)-Q'(x)\}/(t-x).
\end{aligned}
$$

We estimate $A_n(x)$ and $B_n(x)$.

Lemma 4.3 (Kasuga and Sakai [KaS1, Theorems 1.7 and 3.2]). *Let Q satisfy the condition $C(v+1)$. For $|x| \leqslant Da_n$, $D>0$, we have the following estimates*:

(i) $A_n(x) \sim n/a_n$, $|B_n(x)| \leqslant Cn/a_n$,
(ii) *for each odd integer j, $1 \leqslant j \leqslant v-1$, we have*

$$|A_n^{(j)}(x)| \leqslant C|x|n/a_n^{j+2},$$

and for each even integer j, $0 \leqslant j \leqslant v-1$, we have

$$|B_n^{(j)}(x)| \leqslant C|x|n/a_n^{j+2}.$$

Now, we need some preliminaries. By Kasuga and Sakai [KaS1, Theorem 3.3] we have the following differential equation. For any odd integer $n \geqslant 1$

$$
\begin{aligned}
&P_n'' - (Q' + A_n'/A_n)P_n'\\
&\quad + \{(b_nA_nA_{n-1}/b_{n-1}) + B_nB_{n-1} - (xA_{n-1}B_n/b_{n-1})\\
&\quad + B_n' - (A_n'B_n/A_n) - 2r(A_{n-1}/b_{n-1})\}P_n\\
&\quad + 2r(xP_n' - P_n)/x^2 + 2r(B_{n-1} - A_n'/A_n)(P_n/x) = 0,
\end{aligned}
$$

and for any even integer $n \geqslant 2$

$$
\begin{aligned}
&P_n'' - (Q' + A_n'/A_n)P_n' + \{(b_nA_nA_{n-1}/b_{n-1}) + B_nB_{n-1} - (xA_{n-1}B_n/b_{n-1})\\
&\quad + B_n' - (A_n'B_n/A_n)\}P_n + 2r(P_n'/x) + 2rB_n(P_n/x) = 0.
\end{aligned}
$$

We rewrite these differential equations as follows. For any odd integer n,

$$a(x)P_n''(x) + b(x)P_n'(x) + c(x)P_n(x) + D(x) + E(x) = 0, \tag{4.2}$$

where

$$a(x) = A_n(x), \quad b(x) = -Q'(x)A_n(x) - A_n'(x),$$

$$\begin{aligned} c(x) &= \{b_n A_n^2(x)A_{n-1}(x)/b_{n-1}\} + A_n(x)B_n(x)B_{n-1}(x) \\ &\quad - \{xA_n(x)A_{n-1}(x)B_n(x)/b_{n-1}\} + A_n(x)B_n'(x) - A_n'(x)B_n(x) \\ &\quad - 2r\{A_n(x)A_{n-1}(x)/b_{n-1}\} \\ &= c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x) + c_6(x), \end{aligned} \tag{4.3}$$

$$D(x) = 2r\{A_n(x)B_{n-1}(x) - A_n'(x)\}\{P_n(x)/x\},$$
$$E(x) = 2rA_n(x)[\{xP_n'(x) - P_n(x)\}/x^2].$$

For any even integer n

$$a(x)P_n''(x) + b(x)P_n'(x) + c(x)P_n(x) + D(x) + E(x) = 0, \tag{4.4}$$

where

$$a(x) = A_n(x), \quad b(x) = -Q'(x)A_n(x) - A_n'(x),$$

$$\begin{aligned} c(x) &= \{b_n A_n^2(x)A_{n-1}(x)/b_{n-1}\} + A_n(x)B_n(x)B_{n-1}(x) \\ &\quad - \{xA_n(x)A_{n-1}(x)B_n(x)/b_{n-1}\} + A_n(x)B_n'(x) - A_n'(x)B_n(x) \\ &= c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x), \end{aligned} \tag{4.5}$$

$$D(x) = 2rA_n(x)B_n(x)\{P_n(x)/x\}, \quad E(x) = A_n(x)\{P_n'(x)/x\}.$$

By (4.2) and (4.4), for $j = 0, 1, \ldots, v-2$ $(v \geqslant 2)$ we consider the following differential equations:

$$a(x)P_n''(x) + b(x)P_n'(x) + c(x)P_n(x) + D(x) + E(x) = 0, \quad j = 0,$$
$$\begin{aligned} &a(x)P_n'''(x) + \{a'(x) + b(x)\}P_n''(x) + \{b'(x) + c(x)\}P_n'(x) \\ &\quad + c'(x)P_n(x) + D'(x) + E'(x) = 0, \quad j = 1, \end{aligned}$$

$$\begin{aligned} &a(x)P_n^{(j+2)}(x) + \{ja'(x) + b(x)\}P_n^{(j+1)}(x) \\ &\quad + \sum_{s=0}^{j-2} \left\{ \binom{j}{s+2} a^{(s+2)}(x) + \binom{j}{s+1} b^{(s+1)}(x) + \binom{j}{s} c^{(s)}(x) \right\} P_n^{(j-s)}(x) \\ &\quad + \{b^{(j)}(x) + jc^{(j-1)}(x)\}P_n'(x) + c^{(j)}(x)P_n(x) \\ &\quad + D^{(j)}(x) + E^{(j)}(x) = 0, \quad j = 2, 3, \ldots, v-2. \end{aligned}$$

Here, we write simply

$$A_2^{[0]}(x)P_n''(x) + A_1^{[0]}(x)P_n'(x) + A_0^{[0]}(x)P_n(x) \\ + D^{[0]}(x) + E^{[0]}(x) = 0, \quad j = 0,$$

$$A_3^{[1]}(x)P_n'''(x) + A_2^{[1]}(x)P_n''(x) + A_1^{[1]}(x)P_n'(x) \\ + A_0^{[1]}(x)P_n(x) + D^{[1]}(x) + E^{[1]}(x) = 0, \quad j = 1,$$

$$A_{j+2}^{[j]}(x)P_n^{(j+2)}(x) + A_{j+1}^{[j]}(x)P_n^{(j+1)}(x) + \sum_{s=0}^{j} A_{j-s}^{[j]}(x)P_n^{(j-s)}(x) \\ + D^{[j]}(x) + E^{[j]}(x) = 0, \quad j = 2, 3, \ldots, v-2. \tag{4.6}$$

Eq. (4.6) means the following differential equation.

Lemma 4.4 (Kasuga and Sakai [KaS1, Theorem 3.5]). *Let $v \geqslant 2$, and let Q satisfy the condition $C(v+1)$. Then for $j = 0, 1, \ldots, v-2$ we have the following equations*:

$$B_{j+2}^{[j]}(x)P_n^{(j+2)}(x) + B_{j+1}^{[j]}(x)P_n^{(j+1)}(x) + \sum_{s=0}^{j} B_{j-s}^{[j]}(x)P_n^{(j-s)}(x) = 0,$$

where, *for $x_{kn} \neq 0$*,

$$B_{j+2}^{[j]}(x_{kn}) = A_n(x_{kn}) \sim n/a_n,$$

$$|B_{j+1}^{[j]}(x_{kn})| \leqslant CM_n^*(Q; x_{kn})(n/a_n),$$

$$|B_{j-s}^{[j]}(x_{kn})| \leqslant C[\{|x_{kn}|^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle}\} \\ + \{(n/a_n)^{s+2}/|x_{kn}|\}], \quad s = 0, 1, \ldots, j. \tag{4.7}$$

For any odd integer n and $x_{kn} = 0$ we have

$$B_{j+2}^{[j]}(0) = \{1 + 2r/(j+2)\}A_n(0) \sim n/a_n, \quad |B_{j+1}^{[j]}(0)| \leqslant C(n/a_n)^2,$$

$$|B_{j-s}^{[j]}(0)| \leqslant C[\{0^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle}\} + n^2/a_n^{s+3}] \\ \leqslant C(n^3/a_n^{s+3}), \quad s = 0, 1, \ldots, j.$$

Lemma 4.5. *Let $M_n^*(Q; x)$ be defined by (4.1). For $(1/\varepsilon)(a_n/n) \leqslant |x_{kn}| \leqslant \varepsilon a_n$ and n large enough we see*

$$M_n^*(Q; x_{kn}) \leqslant \varepsilon^*(n/a_n), \quad \varepsilon^* = \max(\varepsilon, \varepsilon^{A-1}). \tag{4.8}$$

Proof. By Levin and Lubinsky [LL2, Lemma 5.1(5.3)], we have $Q'(\varepsilon a_n) \leqslant \varepsilon^{A-1} n/a_n$, where A is the constant in (0.1). Therefore, we obtain (4.8). □

After this we write $\varepsilon = \varepsilon^*$ simply. We need Lemma 4.1 again. Let $j = 1, 2, \ldots, \nu$. Then, for $x_{kn} \neq 0$ and $k = 1, 2, \ldots, n$,

$$|P_n^{(j)}(x_{kn})| \leqslant CM_n^*(x_{kn})^{1-\langle j \rangle}(n/a_n)^{j-2+\langle j \rangle}|P_n'(x_{kn})|, \tag{4.9}$$

where C is independent of k and n.

We use Theorem 3.1. Let $r = 1, 2, \ldots, \nu - 1$. Then for $x_{kn} \neq 0$,

$$|(l_{kn}^{\nu})^{(j)}(x_{kn})| \leqslant CM_n^*(x_{kn})^{\langle j \rangle}(n/a_n)^{j-\langle j \rangle} \tag{4.10}$$

for $k = 1, 2, \ldots, n$, where C is independent of k and n.

By Theorem 3.3 we see the following. Let Q satisfy the condition $C(\nu + 1)$. For $i = 1, 2, \ldots, \nu - 1$,

$$e_0(\nu, k, n) = 1, \quad e_i(\nu, k, n) \leqslant C\{M_n^*(Q; x_{kn})\}^{\langle i \rangle}(n/a_n)^{i-\langle i \rangle}. \tag{4.11}$$

Lemma 4.6. *We have an expression*

$$A_n(x_{kn}) = \alpha_n(k)(n/a_n), \quad k = 1, 2, \ldots, n, \tag{4.12}$$

where $\alpha_n(k)$ satisfies $D_1 \leqslant \alpha_n(k) \leqslant D_2$ for positive constants D_1, D_2 independing of n and k. Furthermore, for $j = 0, 1, \ldots, \nu$,

$$\begin{aligned} &B_{j+2}^{[j]}(x_{kn}) = \alpha_n(k)(n/a_n), \\ &|B_j^{[j]}(x_{kn})| = (b_n/b_{n-1})\alpha_n^2(k)\alpha_{n-1}(k)(n/a_n)^3\{1 + \varepsilon_n(j; x_{kn})\}, \end{aligned} \tag{4.13}$$

where there exists $C > 0$ such that

$$|\varepsilon_n(j; x_{kn})| \leqslant C\varepsilon. \tag{4.14}$$

Proof. By Lemma 4.3 we have (4.12). From $B_{j+2}^{[j]}(x_{kn}) = A_{j+2}^{[j]}(x_{kn}) = A_n(x_{kn})$, the first equation in (4.13) is satisfied. By Lemma 4.4 we see that $B_j^{[j]}(x_{kn})$ has the expression

$$B_j^{[j]}(x_{kn}) = \binom{j}{2}a''(x_{kn}) + \binom{j}{1}b'(x_{kn}) + \sum_{i=1}^{6} c_i(x_{kn}) + (n/a_n)^2/|x_{kn}|, \quad x_{kn} \neq 0. \tag{4.15}$$

Here, by (4.3) and (4.5) we see

$$a(x) = A_n(x), \quad b(x) = -Q'(x)A_n(x) - A_n'(x),$$

$$c(x) = c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x) + c_6(x),$$

but if n is odd, then we omit c_6.

First, we deal with the main term $c_1(x_{kn})$. From (4.3) and (4.5) we see

$$\begin{aligned} c_1(x_{kn}) &= (b_n/b_{n-1})A_n^2(x_{kn})A_{n-1}(x_{kn}) \\ &= (b_n/b_{n-1})\alpha_n^2(k)\alpha_{n-1}(k)(n/a_n)^3\{1 + \varepsilon_n'(j; x_{kn})\}. \end{aligned}$$

By Kasuga and Sakai [KaS1, Proof of Theorem 3.4] we see the following:

$$|a''(x_{kn})| \leqslant C(n/a_n^3), \quad |b'(x_{kn})| \leqslant C(n^2/a_n^3),$$
$$|c_2(x_{kn})| \leqslant C\varepsilon^2(n/a_n)^3, \quad |c_3(x_{kn})| \leqslant C\varepsilon^2(n/a_n)^3,$$
$$|c_4(x_{kn})| \leqslant C(n^2/a_n^3), \quad |c_5(x_{kn})| \leqslant C\varepsilon^2(n^2/a_n^3),$$
$$|c_6(x_{kn})| \leqslant C(n^2/a_n^3) \quad (\text{let } c_6 = 0 \text{ if } n \text{ is even}).$$

Noting (4.15), for n large enough, we have (4.14)

$$|\varepsilon_n(j; x_{kn})| \leqslant C\varepsilon.$$

Therefore, the proof of Lemma 4.6 is complete. □

Remark 4.7. For $Q(x) = |x|^{2m}, m = 1, 2, 3, \ldots$, we have the following.

$$\alpha_n(k) = \alpha_n(Q) = 2m^{(4m-1)/2m}\binom{2m-2}{m-1}\beta^{2m-1},$$

where β is the Freud's constant (see [KS1]).

Using the above Lemma 4.6, we can estimate the lower bound for $P_n^{(2s+1)}(x_{kn})$, $2s + 1 \leqslant v$.

Lemma 4.8. *Let $s = 1, 2, \ldots, (v-1)/2$, $0 < \varepsilon$ (small enough), and $(1/\varepsilon)(a_n/n) \leqslant |x_{kn}| \leqslant \varepsilon a_n$. If we set*

$$P_n^{(2s+1)}(x_{kn}) = (-1)^s \beta_n^s(k)(n/a_n)^{2s}\{1 + \zeta_n(s; x_{kn})\}P_n'(x_{kn}),$$
$$\beta_n(k) = (b_n/b_{n-1})\alpha_n(k)\alpha_{n-1}(k), \tag{4.16}$$

then for n large enough,

$$|\zeta_n(s; x_{kn})| \leqslant C\varepsilon, \tag{4.17}$$

where C is independent of n, x_{kn} and ε, and may depend on s and Q.

Remark 4.9. From $b_n \sim b_{n-1}$ we see that there exist positive constants C_1, C_2 independent of n and k such that

$$C_1 \leqslant \beta_n(k) \leqslant C_2. \tag{4.18}$$

Proof of Lemma 4.8. Let $j = 0, 1, \ldots, v$. First, by (4.7) we note that

$$|B_{j+2}^{[j]}(x_{kn})| \geqslant C(n/a_n), \tag{4.19}$$

$$|B_{j+1}^{[j]}(x_{kn})| \leqslant C\varepsilon(n/a_n)^2, \tag{4.20}$$

$$|B_{j-s}^{[j]}(x_{kn})| \leqslant C\varepsilon^{\langle s \rangle}\{(n^3/a_n^{3+s}) + (n/a_n)^{s+2}\}, \quad s = 1, 2, \ldots, j-1 \tag{4.21}$$

for $(1/\varepsilon)(a_n/n) \leqslant |x_{kn}| \leqslant \varepsilon a_n$, where C is independent of n, k and ε, and may depend on j and $Q(x)$. By (4.8) and (4.9),

$$P_n^{(j)}(x_{kn}) \leqslant C\varepsilon^{1-\langle j \rangle}(n/a_n)^{j-1}|P_n'(x_{kn})|, \quad j = 1, 2, \ldots, \nu \tag{4.22}$$

for $(1/\varepsilon)(a_n/n) \leqslant |x_{kn}| \leqslant \varepsilon a_n$, where C is independent of n, k and ε. By (4.13) and (4.14) we see that for $j = 0, 1, \ldots, \nu$

$$\begin{aligned} &-B_j^{[j]}(x_{kn})/B_{j+2}^{[j]}(x_{kn}) \\ &\quad = (-1)\beta_n(k)(n/a_n)^2\{1 + \rho_n(j; x_{kn})\}, \ |\rho_n(j; x_{kn})| \leqslant C\varepsilon, \end{aligned} \tag{4.23}$$

for $(1/\varepsilon)(a_n/n) \leqslant |x_{kn}| \leqslant \varepsilon a_n$, where C is independent of n, k and ε.

Now, we show (4.16) and (4.17) by induction on s. Let $s = 1$. It follows from Lemma 4.4 that

$$\begin{aligned} P_n^{(3)}(x_{kn}) = &-\{B_2^{[1]}(x_{kn})/B_3^{[1]}(x_{kn})\}P_n''(x_{kn}) \\ &-\{B_1^{[1]}(x_{kn})/B_3^{[1]}(x_{kn})\}P_n'(x_{kn}). \end{aligned}$$

By (4.18), (4.19) and (4.21), the first term on the right-hand side of the above equality is bounded by $C\varepsilon^2(n/a_n)^2|P_n'(x_{kn})|$. The second term is estimated by (4.22). These lead to (4.16), (4.17) with $s = 1$

$$\begin{aligned} P_n^{(3)}(x_{kn}) &= \{(-1)\beta_n(k)(1 + \rho_{kn}) + C\varepsilon(a_n/n)\}(n/a_n)^2P_n'(x_{kn}) \\ &= (-1)\{\beta_n(k)(1 + \zeta_n(1; x_{kn}))\}(n/a_n)^2P_n'(x_{kn}), \quad |\zeta_n(1; x_{kn})| \leqslant C\varepsilon \end{aligned}$$

for ε small enough and n large enough.

We suppose (4.16) and (4.17) until $s - 1(\geqslant 1)$ holds. From the expression of Lemma 4.4 it follows that

$$\begin{aligned} P_n^{(2s+1)} = &-(B_{2s}/B_{2s+1})P_n^{(2s)} - (B_{2s-1}/B_{2s+1})P_n^{(2s-1)} \\ &-(B_{2s-2}/B_{2s+1})P_n^{(2s-2)} - \cdots - (B_1/B_{2s+1})P_n^{(1)}, \end{aligned} \tag{4.24}$$

where B_j and $P_n^{(j)}$ stand for $B_j^{[2s-1]}(x_{kn})$ and $P_n^{(j)}(x_{kn})$, respectively. By the assumption of induction and (4.22), we see that the second term on the right-hand side of (4.23) has an estimate

$$\begin{aligned} &-(B_{2s-1}/B_{2s+1})P_n^{(2s-1)}(x_{kn}) \\ &\quad = (-1)\beta_n(k)(n/a_n)^2\{1 + \rho_n(2s-1; x_{kn})\}(-1)^{s-1}\beta_n^{s-1}(k)(n/a_n)^{2(s-1)} \\ &\qquad \times \{1 + \zeta_n(s-1; x_{kn})\}P_n'(x_{kn}) \\ &\quad = (-1)^s\beta_n^s(k)(n/a_n)^{2s}\{1 + \rho_n'(2s+1; x_{kn})\}P_n'(x_{kn}), \end{aligned}$$

where

$$\begin{aligned} &\rho_n'(2s+1; x_{kn}) \\ &\quad = \rho_n(2s-1; x_{kn}) + \zeta_n(s-1; x_{kn}) + \rho_n(2s-1; x_{kn})\zeta_n(s-1; x_{kn}). \end{aligned}$$

Then we have $|\rho_n'(2s+1;x_{kn})|\leqslant C\varepsilon$. Combining (4.18)–(4.21), we easily see that the other terms on the right-hand side of (4.23) are bounded by $C(n/a_n)^{2s}(\varepsilon^2+a_n^{-2})|P_n'(x_{kn})|$. Now, if we take n large enough as $a_n^{-1}<\varepsilon$, then we obtain (4.16) and (4.17)

$$P_n^{(2s+1)}(x_{kn})=(-1)^s\beta_n^s(k)(n/a_n)^{2s}\{1+\zeta_n(s;x_{kn})\}P_n'(x_{kn}),$$
$$|\zeta_n(1;x_{kn})|\leqslant C\varepsilon,$$

where $\zeta_n(s;x_{kn})=\rho_n'(2s+1;x_{kn})\zeta_n'(s;x_{kn})$. □

We need more refined estimate of $(l_{kn}^{\nu})^{(2j)}(x_{kn})$. Let $\phi_j(1)=(2j+1)^{-1}$, $j=0,1,2,\ldots$. Let $0<\varepsilon<1$, and suppose $(1/\varepsilon)(a_n/n)\leqslant|x_{kn}|\leqslant\varepsilon a_n$. From $x_{kn}\neq 0$, we see

$$\begin{aligned}l_{kn}(x)&=P_n(x)/\{(x-x_{kn})P_n'(x_{kn})\}\\&=\{1/P_n'(x_{kn})\}\sum_{i=1}^{n}\{P_n^{(i)}(x_{kn})/i!\}(x-x_{kn})^{i-1}.\end{aligned}$$

So we have

$$l_{kn}^{(2j)}(x_{kn})=P_n^{(2j+1)}(x_{kn})/\{(2j+1)P_n'(x_{kn})\}.$$

Therefore, from this and Lemma 4.8, we have

$$\begin{aligned}&l_{kn}^{(2j)}(x_{kn})=(-1)^j\phi_j(1)\beta_n^j(k)(n/a_n)^{2j}\{1+\zeta_n(1,j;x_{kn})\},\\&|\zeta_n(1,j;x_{kn})|\leqslant C\varepsilon,\quad j=0,1,\ldots,\nu,\end{aligned}\tag{4.25}$$

where $\zeta_n(1,j;x_{kn})=\zeta_n(j;x_{kn})$ or $j\geqslant 1$, $\zeta_n(1,0;x_{kn})=0$, and C is independent of n, x_{kn} and ε, and may depend on j and Q. By induction on ν, we can estimate $(l_{kn}^{\nu})^{(2j)}(x_{kn})$.

Lemma 4.10 (Cf. Kasuga and Sakai [KS1, Lemma 10]). *Let $0<\varepsilon<1$, and suppose $(1/\varepsilon)(a_n/n)\leqslant|x_{kn}|\leqslant\varepsilon a_n$. Then, for $\nu=1,2,3,\ldots$, there exists uniquely a sequence $\{\phi_j(\nu)\}_{j=0}^{\infty}$ of positive numbers and $\zeta_n(\nu,j;x_{kn})$ such that*

$$\begin{aligned}&(l_{kn}^{\nu})^{(2j)}(x_{kn})=(-1)^j\phi_j(\nu)\beta_n^j(k)(n/a_n)^{2j}\{1+\zeta_n(\nu,j;x_{kn})\},\\&|\zeta_n(\nu,j;x_{kn})|\leqslant C\varepsilon,\quad j=0,1,\ldots,\nu,\end{aligned}\tag{4.26}$$

where C is independent of n, x_{kn} and ε, and may depend on ν, j and Q.

Proof. The case of $v = 1$ follows from (4.24). Suppose that for the case of $v-1$ the lemma holds. We have

$$\begin{aligned}(l_{kn}^{v})^{(2j)}(x_{kn}) &= \sum_{i=0}^{2j}\binom{2j}{i}(l_{kn}^{v-1})^{(i)}(x_{kn})l_{kn}^{(2j-i)}(x_{kn}) \\ &= \sum_{r=0}^{j}\binom{2j}{2r}(l_{kn}^{v-1})^{(2r)}(x_{kn})l_{kn}^{(2j-2r)}(x_{kn}) \\ &\quad + \sum_{r=1}^{j}\binom{2j}{2r-1}(l_{kn}^{v-1})^{(2r-1)}(x_{kn})l_{kn}^{(2j-2r+1)}(x_{kn}).\end{aligned}$$

It follows from (4.8) and (4.10) that

$$|(l_{kn}^{v})^{(2t-1)}(x_{kn})| \leqslant C\varepsilon(n/a_n)^{2t-1}, \quad t = 1, 2, 3, \ldots,$$

therefore, the second sum on the right-hand side of the above equality is bounded by $C\varepsilon(n/a_n)^{2t}$. By (4.20) and the assumption of induction, the first sum $\sum_{i=0}^{j}$ is estimated as

$$\begin{aligned}\sum_{r=0}^{j} &= \sum_{r=0}^{j}\binom{2j}{2r}(-1)^r\phi_r(v-1)\beta_n^r(k)(n/a_n)^{2r}\{1+\zeta_n(v-1,j;x_{kn})\} \\ &\quad \times(-1)^{j-r}\phi_{j-r}(1)\beta_n^{j-r}(k)(n/a_n)^{2(j-r)}\{1+\zeta_n(1,j-r;x_{kn})\} \\ &= \sum_{r=0}^{j}\{1/(2j-2r+1)\}\binom{2j}{2r}(-1)^j\phi_r(v-1)\beta_n^j(k)(n/a_n)^{2j} \\ &\quad \times\{1+\tau_n(v,j,r;x_{kn})\},\end{aligned}$$

where

$$\begin{aligned}\tau_n(v,j,r;x_{kn}) &= \zeta_n(v-1,j;x_{kn}) + \zeta_n(1,j-r;x_{kn}) \\ &\quad + \zeta_n(v-1,j;x_{kn})\zeta_n(1,j-r;x_{kn}),\end{aligned}$$

$$|\tau_n(v,j,r;x_{kn})| \leqslant C\varepsilon.$$

If we put, for $j = 0, 1, 2, \ldots,$

$$\phi_j(v) = \sum_{r=0}^{j}\{1/(2j-2r+1)\}\binom{2j}{2r}\phi_r(v-1),$$

$$\zeta_n(v,j;x_{kn}) = \sum_{r=0}^{j}\{1/(2j-2r+1)\}\binom{2j}{2r}\phi_r(v-1)\tau_n(v,j,r;x_{kn}), \tag{4.27}$$

then $\{\phi_j(v)\}_{j=0}^{\infty}$ and $\{\zeta_n(v,j,r;x_{kn})\}_{j=0}^{\infty}$ satisfy the required conditions (4.25). □

We rewrite relation (4.26) in the form

$$\phi_0(v) = 1, \quad v = 1, 2, 3, \ldots,$$

$$\phi_j(v) - \phi_j(v-1) = \{1/(2j+1)\} \sum_{r=0}^{j-1} \binom{2j+1}{2r} \phi_r(v-1),$$
$$j = 1, 2, 3, \ldots, \; v = 2, 3, 4, \ldots \; .$$

Now, for every j we will introduce an auxiliary polynomial determined by $\{\phi_j(v)\}_{j=1}^{\infty}$ as the following lemma.

Lemma 4.11 (Kanjin and Sakai [KS1, Lemma 11]). (i) *For $j = 0, 1, 2, \ldots,$ there exists a unique polynomial $\Psi_j(y)$ of degree j such that $\Psi_j(v) = \phi_j(v)$, $v = 1, 2, 3, \ldots$.*
(ii) $\Psi_0(y) = 1$, *and* $\Psi_j(0) = 0$, $j = 1, 2, 3, \ldots$.

Since $\Psi_j(y)$ is a polynomial of degree j, we can replace $\phi_j(v)$ in (4.27) with $\Psi_j(y)$, that is,

$$\Psi_j(y) = \sum_{r=0}^{j} \{1/(2j-2r+1)\} \binom{2j}{2r} \Psi_r(y-1), \quad j = 0, 1, 2, \ldots, \tag{4.28}$$

for an arbitrary y and $j = 0, 1, 2, \ldots$. We use the notation $F_{kn}(x, y) = \{l_{kn}(x)\}^y$ which coincides with $l_{kn}^y(x)$ if y is an integer. Since $l_{kn}(x_{kn}) = 1$, we have $F_{kn}(x, t) > 0$ for x in a neighbourhood of x_{kn} and an arbitrary real number y.

We will show that $(\partial/\partial x)^j F_{kn}(x_{kn}, y)$ is a polynomial of degree at most j with respect to y for $j = 0, 1, 2, \ldots,$ where $(\partial/\partial x)^j F_{kn}(x_{kn}, y)$ is the jth partial derivative of $F_{kn}(x, y)$ with respect to x at (x_{kn}, y). We prove these facts by induction on j. For $j = 0$ it is trivial. Suppose that it holds for $j \geqslant 0$. To simplify the notation, let $F(x) = F_{kn}(x, y)$ and $l(x) = l_{kn}(x)$ for a fixed y. Then $F'(x)l(x) = yl'(x)F(x)$. By Leibniz's rule, we easily see that

$$F^{(j+1)}(x_{kn}) = -\sum_{s=0}^{j-1} \binom{j}{s} F^{(s+1)}(x_{kn}) l^{(j-s)}(x_{kn})$$
$$+ \; y \sum_{s=0}^{j} \binom{j}{s} l^{(s+1)}(x_{kn}) F^{(j-s)}(x_{kn}),$$

which shows that $F^{(j+1)}(x_{kn})$ is a polynomial of degree at most $j+1$ with respect to y.

Let $P_{kn}^{[j]}(y)$ be defined by

$$(\partial/\partial x)^{2j} F_{kn}(x_{kn}, y) = (-1)^j \beta_n^j(k)(n/a_n)^{2j} \Psi_j(y) + P_{kn}^{[j]}(y), \quad j = 0, 1, 2, \ldots \; .$$

Then $P_{kn}^{[j]}(y)$ is a polynomial of degree at most $2j$. We have the following.

By Lemma 4.10 we have the following.

Lemma 4.12 (Kanjin and Sakai [KS1, Lemma 12]). *Let* $j = 0, 1, 2, \ldots,$ *and let* M *be a positive constant. If* $(1/\varepsilon)(a_n/n) \leqslant |x_{kn}| \leqslant \varepsilon a_n$, $0 < \varepsilon$ *(small enough), and* $|y| \leqslant M$, *then,*

(i) $|(\partial/\partial y)^s P_{kn}^{[j]}(y)| \leqslant C\varepsilon(n/a_n)^{2j}$, $s = 0, 1$,
and
(ii) $|(\partial/\partial x)^{2j+1} F_{kn}(x_{kn}, y)| \leqslant C\varepsilon(n/a_n)^{2j+1}$,
where C *is independent of* n, k *and* ε, *and may depend on* j, M *and* Q.

By (i) of the above Lemma 4.12, we can prove the following lemma which plays an essential role in estimating the lower bound of $e_{v-1}(v, k, n)$.

Lemma 4.13 (Cf. Kanjin and Sakai [KS1, Lemma 13]). *If* $y < 0$, *then* $\Psi_j(y) \neq 0$ *for* $j = 0, 1, 2, \ldots$.

Proof. Since $\Psi_0(y) = 1$, we may assume $j \geqslant 1$. Since $\Psi_j(0) = 0$, $\Psi_j(y)$ has an expression

$$\Psi_j(y) = \sum_{i=1}^{j} (-1)^{j-i} c_i(j) y^i, \quad j = 1, 2, 3, \ldots . \tag{4.30}$$

Then it is enough to show that $c_i(j) > 0$, $j = 1, 2, 3, \ldots$. Because if $y = -u$, $u > 0$, then $\Psi_j(-u) = (-1)^j \sum_{i=1}^{j} c_i(j) u^i \neq 0$.

We will first show that $c_1(j) > 0$, $j = 1, 2, 3, \ldots$. It follows from (4.25) and $(-1)^{j-1} c_1(j) = (d/dy)\Psi_j(0)$ that

$$-\beta_n^j(k)(n/a_n)^{2j} c_1(j) = (d/dy)\{(\partial/\partial x)^{2j} F_{kn}(x_{kn}, y) - P_{kn}^{[j]}(y)\}_{y=0}$$

(see (4.29)). We have

$$\begin{aligned} (d/dy)\{(\partial/\partial x)^{2j} F_{kn}(x_{kn}, y)\}_{y=0} &= (d/dx)^{2j}\{(\partial/\partial y) F_{kn}(x, 0)\}_{x=x_{kn}} \\ &= (d/dx)^{2j} \log\{|l_{kn}(x)|\}_{x=x_{kn}} \\ &= -\ (2j-1)! \sum_{s \neq k} \{1/(x_{kn} - x_{sn})^{2j}\}. \end{aligned}$$

Here, we used the expression $l_{kn}(x) = P_n(x)/\{(x - x_{kn}) P_n'(x_{kn})\}$. Therefore, we have

$$c_1(j) = \beta_n^{-j}(k)(n/a_n)^{-2j} \left[(2j-1)! \sum_{s \neq k} \{1/(x_{kn} - x_{sn})^{2j}\} + (d/dy) P_{kn}^{[j]}(0) \right].$$

From Lemma 4.12(i) it follows that $|(d/dy) P_{kn}^{[j]}(0)| \leqslant C\varepsilon(n/a_n)^{2j}$ for a certain number k as $(1/\varepsilon)(n/a_n) \leqslant |x_{kn}| \leqslant \varepsilon a_n$, where C is a positive constant independent of n. From this and $x_{k-1,n} - x_{k+1,n} \sim (n/a_n)$ (see [KaS1, Theorem 1.4]), we have

$$\begin{aligned} c_1(j) &\geqslant \beta_n^{-j}(k)(n/a_n)^{-2j}\{C(2j-1)!(n/a_n)^{2j} - C\varepsilon(n/a_n)^{2j}\} \\ &\geqslant \{C(2j-1)!\beta_n^{-j}(k) - C\varepsilon\}. \end{aligned}$$

Letting $\varepsilon \to 0$, we see that $c_1(j) > 0$.

Next, we treat the other coefficients. We see that

$$
\begin{aligned}
&(l_{kn}^{2\mu})^{(2j+2)}(x_{kn}) \\
&\quad = \sum_{r=0}^{j+1} \binom{2j+2}{2r} (l_{kn}^{\mu})^{(2r)}(x_{kn})(l_{kn}^{\mu})^{(2j+2-2r)}(x_{kn}) \\
&\qquad + \sum_{r=1}^{j+1} \binom{2j+2}{2r-1} (l_{kn}^{\mu})^{(2r-1)}(x_{kn})(l_{kn}^{\mu})^{(2j+3-2r)}(x_{kn}), \quad \mu = 1, 2, 3, \ldots .
\end{aligned}
$$

From (4.25), it follows that the leading term on the left-hand side of the equation is

$$(-1)^{j+1}\phi_{j+1}(2\mu)\beta_n^{j+1}(k)(n/a_n)^{2(j+1)}.$$

The leading term of the first sum on the right-hand side is

$$\sum_{r=0}^{j+1} \binom{2j+2}{2r} (-1)^{j+1}\phi_r(\mu)\phi_{j+1-r}(\mu)\beta_n^{j+1}(k)(n/a_n)^{2(j+1)}.$$

Since $|(l_{kn}^{\mu})^{(2t-1)}(x_{kn})| \leqslant C\varepsilon(n/a_n)^{2t-1}$, $t = 1, 2, \ldots$. Therefore, we have

$$\phi_{j+1}(2\mu) = \sum_{r=0}^{j+1} \binom{2j+2}{2r} \phi_r(\mu)\phi_{j+1-r}(\mu)$$

as $\varepsilon \to 0$, and therefore,

$$\phi_{j+1}(2\mu) - 2\phi_{j+1}(\mu) = \sum_{r=1}^{j} \binom{2j+2}{2r} \phi_r(\mu)\phi_{j+1-r}(\mu), \quad \mu = 1, 2, 3, \ldots .$$

This leads to

$$\Psi_{j+1}(2y) - 2\Psi_{j+1}(y) = \sum_{r=1}^{j} \binom{2j+2}{2r} \Psi_r(y)\Psi_{j+1-r}(y). \tag{4.31}$$

We replace

$$\Psi_{j+1}(y) = \sum_{i=1}^{j+1} (-1)^{j+1-i} c_i(j+1) y^i.$$

By (4.31) we have

$$\sum_{i=1}^{j+1} (-1)^{j+1-i}(2^i - 2)c_i(j+1)y^i = \sum_{r=1}^{j} \binom{2j+2}{2r} \Psi_r(y)\Psi_{j+1-r}(y).$$

If we assume $c_i(j) > 0$, $i = 1, 2, \ldots, j$, then we see that the right-hand side of the equation is a polynomial of degree $j+1$, whose coefficients are alternating. Therefore, we have $(2^i - 2)c_i(j+1) > 0$, which implies $c_i(j+1) > 0$, $i = 2, \ldots, j+1$. This completes the proof since we have already obtained $c_1(j) > 0$, $j = 1, 2, 3, \ldots$. □

Proof of Theorem 3.6. Let $0<\varepsilon$ (small enough). For $v=1,2,3,\ldots$, we define $\eta_{kn}(v;s)$ by (1.1), that is

$$e_{2s}(v,k,n)=(-1)^s\{1/(2s)!\}\Psi_s(-v)\beta_n^s(k)(n/a_n)^{2s}\{1+\eta_{kn}(v;s)\}.$$

Then, we will show $|\eta_{kn}(v;s)|\leqslant C\varepsilon$ for k and $(1/\varepsilon)(a_n/n)\leqslant|x_{kn}|\leqslant\varepsilon a_n$ and $s=0,1,2,\ldots,(v-1)/2$, where C is independent of n, k and ε, and may depend on v, s and Q.

We prove (3.1) and (3.2) by induction on s. By the definition of $h_{kn}(v;x)$, we have

$$e_0(v,k,n)=1,$$

$$\begin{aligned} e_j(v,k,n)=&-(1/j!)\sum_{r=0}^{j-1}\{j!/(j-r)!\}\\ &\times e_r(v,k,n)(l_{kn}^v)^{(j-r)}(x_{kn}),\quad j=1,2,\ldots,v-1. \end{aligned} \tag{4.32}$$

By $e_0(v,k,n)=1$ and $\Psi_0(y)=1$, (3.1) holds for $s=0$. From (4.32), we write $e_{2s}(v,k,n)$ in the form

$$\begin{aligned} &e_{2s}(v,k,n)\\ &=-\{1/(2s)!\}\left[\sum_{r=0}^{s-1}\{(2s)!/(2s-2r)!\}e_{2r}(v,k,n)(l_{kn}^v)^{(2s-2r)}(x_{kn})\right.\\ &\quad\left.+\sum_{r=1}^{s}\{(2s)!/(2s-2r+1)!\}e_{2r-1}(v,k,n)(l_{kn}^v)^{(2s-2r+1)}(x_{kn})\right]. \end{aligned}$$

We have $|(l_{kn}^v)^{(2s-2r+1)}(x_{kn})|\leqslant C\varepsilon(n/a_n)^{2s-2r+1}$ by (4.8), (4.10) and $|e_{2r-1}(v,k,n)|\leqslant C\varepsilon(n/a_n)^{2r-1}$ (see (4.8) and (4.11)). The second sum $\sum_{r=1}^{s}$ is bounded by $C\varepsilon^2(n/a_n)^{2s}$. For the first sum $\sum_{r=0}^{s-1}$ we have the following. By (3.1), (3.2) and Lemma 4.10,

$$\begin{aligned} \sum_{r=0}^{s-1}=&\sum_{r=0}^{s-1}(-1)^r\{(2s)!/(2s-2r)!\}\{1/(2r)!\}\Psi_r(-v)\beta_n^r(k)(n/a_n)^{2r}\\ &\times\{1+\eta_{kn}(v,r)\}(-1)^{s-r}\phi_{s-r}(v)\beta_n^{s-r}(k)(n/a_n)^{2(s-r)}\\ &\times\{1+\zeta_{kn}(v,s-r,x_{kn})\}\\ =&(-1)^s\beta_n^s(k)(n/a_n)^{2s}\sum_{r=0}^{s-1}\binom{2s}{2r}\Psi_r(-v)\phi_{s-r}(v)(1+\lambda_{kn}(r,s)), \end{aligned}$$

where $\lambda_{kn}(r,s)=\eta_{kn}(v,r)+\zeta_{kn}(v,s-r,x_{kn})+\zeta_{kn}(v,s-r,x_{kn})\eta_{kn}(v,r)$. We set

$$\eta_{kn}(v,s)=\sum_{r=0}^{s-1}\binom{2s}{2r}\Psi_r(-v)\phi_{s-r}(v)\lambda_{kn}(r,s),$$

then by $|\lambda_{kn}(r,s)| \leqslant C\varepsilon$ we see $|\eta_{kn}(v,s)| \leqslant C\varepsilon$. Therefore, by Lemma 4.10 and the assumption of induction, it is enough to show

$$\sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-v)\phi_{s-r}(v) = 0, \quad s = 1,2,3,\ldots, \; v = 1,2,3,\ldots .$$

Let $C_s(y) = \sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-y)\Psi_{s-r}(y)$. It suffices to show that $C_s(v) = 0$, $s = 1,2,3,\ldots$, $v = 1,2,3,\ldots$. We have

$$\begin{aligned} 0 = (l_{kn}^{-1+1})^{(2s)}(x_{kn}) &= \sum_{i=0}^{2s} \binom{2s}{i} (l_{kn}^{-1})^{(i)}(x_{kn}) l_{kn}^{(2s-i)}(x_{kn}) \\ &= \sum_{r=0}^{s} \binom{2s}{2r} (\partial/\partial x)^{2r} F_{kn}(x_{kn},-1) l_{kn}^{(2s-2r)}(x_{kn}) \\ &\quad + \sum_{r=0}^{s-1} \binom{2s}{2r+1} (\partial/\partial x)^{2r+1} F_{kn}(x_{kn},-1) l_{kn}^{(2s-2r-1)}(x_{kn}) \end{aligned}$$

for every s. By (4.25), (4.28) and Lemma 4.12(i), we see that the first sum $\sum_{r=0}^{s}$ has the form

$$\sum_{r=0}^{s} = (-1)^s \beta_n^s(k)(n/a_n)^{2s} \sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-1)\phi_{s-r}(1) + \xi_n (n/a_n)^{2s},$$

where $|\xi_n| \leqslant C\varepsilon$. By (4.10) and Lemma 4.12(ii), the second sum $\sum_{r=0}^{s-1}$ is bounded by $C\varepsilon(n/a_n)^{2s}$. Therefore, letting $\varepsilon \to 0$, we see that

$$0 = \sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-1)\Psi_{s-r}(1) = C_s(1)$$

for every s. Suppose $C_s(v) = 0$ for every s. We will show that $C_s(v+1) = 0$ for every s. Using (4.27) and changing the order of summation, we have

$$\begin{aligned} &C_s(v+1) \\ &\quad = \sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-v-1) \sum_{p=0}^{s-r} \{1/(2s-2r-2p+1)\} \binom{2s-2r}{2p} \Psi_p(v) \\ &\quad = \sum_{p=0}^{s} \left[\sum_{r=0}^{s-p} \{1/(2s-2r-2p+1)\} \binom{2s-2r}{2p} \binom{2s}{2r} \Psi_r(-v-1) \right] \Psi_p(v). \end{aligned}$$

By the relation $\binom{2s-2r}{2p}\binom{2s}{2r} = \binom{2s}{2p}\binom{2s-2p}{2r}$ and (4.27), we have

$$\begin{aligned} &\sum_{r=0}^{s-p} \{1/(2s-2r-2p+1)\} \binom{2s-2r}{2p} \binom{2s}{2r} \Psi_r(-v-1) \\ &\quad = \binom{2s}{2p} \Psi_{s-p}(-v), \end{aligned}$$

with leading to $C_s(v+1) = C_s(-v)$. Since we easily see $C_s(-v) = C_s(v)$, we finish proving. The positiveness $(-1)^j \Psi_j(-v) > 0$, $j = 0, 1, 2, \ldots$, $v = 1, 2, 3, \ldots$, are easily obtained by (4.30). □

5. Applications

In this section we report some interesting applications of results in the previous sections. We suppose again $r \geqslant 0$ in (0.3). We define the moduli of continuity of $f \in C(\mathbf{R})$ by

$$\omega(f, [a,b]; h) = \max_{|x_1 - x_2| \leqslant h,\ x_1, x_2 \in [a,b]} |f(x_1) - f(x_2)|, \quad h > 0$$

and

$$\omega(f, \mathbf{R}; h) = \max_{|x_1 - x_2| \leqslant h,\ x_1, x_2 \in \mathbf{R}} |f(x_1) - f(x_2)|, \quad h > 0.$$

Theorem 5.1. *Let Q satisfy the condition $C(v)$, and let $v = 1, 2, 3, \ldots$. If $f \in C(\mathbf{R})$ is uniformly continuous function on $\mathbf{R}$, then we have*

$$\sup_{x \in \mathbf{R}} W_{rQ}^{v}(x)(1+|x|)^{-v\eta/6} |L_n(v, f; x) - f(x)| \leqslant C \log(1+n)\, \omega(f, \mathbf{R}; a_n/n),$$

where

$$\sup_{0 \leqslant u < \infty} uQ'(u)/Q(u) = \eta_Q, \quad \eta_Q \leqslant \eta.$$

Remark 5.2. If $\lim_{n \to \infty} \log(1+n)\omega(f, \mathbf{R}; a_n/n) = 0$ (for example, $f \in \mathrm{Lip}_\alpha(\mathbf{R}) = \{f; |f(x+h) - f(x)| \leqslant C|h|^\alpha\}$), then

$$\lim_{n \to \infty} \sup_{x \in \mathbf{R}} W_{rQ}^{v}(x)(1+|x|)^{-v\eta/6} |L_n(v, f; x) - f(x)| = 0.$$

Theorem 5.3 (Cf. Kanjin and Sakai [KS1]). *Let $v \geqslant 1$ be an odd integer, and let Q satisfy the condition $C(v+1)$. Then there is a function $f \in \mathbf{C}(\mathbf{R})$ such that for any fixed constant $M > 0$,*

$$\limsup_{n \to \infty} \max_{-M \leqslant x \leqslant M} |L_n(v, f; x)| = \infty.$$

Theorem 5.4 (Cf. Kanjin and Sakai [KS2]). *Let Q satisfy the condition $C(v)$, and let I be any compact interval.*

(i) *Let $v - 1 = l$, and $N \geqslant l$. If $f \in C^{(N)}(\mathbf{R})$ satisfies*

$$\lim_{h \to 0} \omega(f^{(N)}, \mathbf{R}; h) \log(h) = 0,$$

then we have

$$\lim_{n \to \infty} \max_{x \in I} W_{rQ}^{v}(x) |L_n^{(j)}(v-1, v, f; x) - f^{(j)}(x)| = 0, \quad 0 \leqslant j \leqslant N\{1 - (1/(v+2))\}.$$

(ii) *Let* $v-1>l$. *If* $f\in C^{(l)}(\mathbf{R})$ *satisfies*

$$\lim_{h\to 0}\omega(f^{(l)},\mathbf{R};h)\log(h)=0,$$

then we have

$$\lim_{n\to\infty}\max_{x\in I} W_{rQ}^{v}(x)|L_n^{(j)}(l,v,f;x)-f^{(j)}(x)|=0,\quad 0\leqslant j\leqslant l\{1-(1/(v+2))\}.$$

We will show only Theorem 5.1. The proofs of other theorems are completed by the same line of proofs as [KS1] or [KS2].

Lemma 5.5. *Let* $v\geqslant 2$, *and let* $f\in C(\mathbf{R})$ *be uniformly continuous on* $\mathbf{R}$. *Then we have*

$$W_{rQ}^{v}(x)|f(x)|\leqslant C\omega(f,\mathbf{R};a_n/n),\quad |x|\geqslant a_n.$$

Proof. First, we show that $W_{rQ}^{1/2}(x)|f(x)|$ is bounded on $\mathbf{R}$. In fact, if it is not true, then we see that there exists a sequence $\{x_k\}_{k=1}^{\infty}, 0<x_1<x_2<x_3<\dots,\ x_{k+1}-x_k\geqslant 1$, such that $W_{rQ}^{1/2}(x_k)|f(x_k)|=\mu(x_k)>1$. For simplicity, we suppose that $f(x_k)>0$. We may consider that $\mu(x_k)$ is increasing, then

$$\begin{aligned}f(x_{k+1})-f(x_k)&=\mu(x_{k+1})W_{rQ}^{-1/2}(x_{k+1})-\mu(x_k)W_{rQ}^{-1/2}(x_k)\\&\geqslant \mu(x_1)\{W_{rQ}^{-1/2}(x_{k+1})-W_{rQ}^{-1/2}(x_k)\}.\end{aligned}$$

Since $f(x)$ is continuous, we see that for any fixed $h,\ 0<h<1$ there exists a sequence $\{x_k(h)\}_{k=1}^{\infty}$ such that $x_k\leqslant x_k(h)$ and

$$\begin{aligned}\{f(x_k(h)+h)-f(x_k(h))\}/h&=\{f(x_{k+1})-f(x_k)\}/(x_{k+1}-x_k)\\&\geqslant \mu(x_1)\{W_{rQ}^{-1/2}(x_{k+1})-W_{rQ}^{-1/2}(x_k)\}/(x_{k+1}-x_k)\\&\geqslant C(W_{rQ}^{-1/2})'(x_k),\end{aligned}$$

where C is a positive constant. Here, for k large enough we have $h(W_{rQ}^{-1/2})'(x_k)\geqslant 1$. Then we see $f(x_k(h)+h)-f(x_k(h))\geqslant C$, where C is a positive constant independent of h. But for h small enough this contradicts the uniformly continuity.

Now, since $Q(a_n)\sim n$ (see [LL2, Lemma 5.2]), we have for $|x|\geqslant a_n$

$$W_{rQ}^{v}(x)|f(x)|\leqslant CW_{rQ}^{1/2}(a_n)\leqslant Ca_n/n\leqslant C\omega(f,\mathbf{R};a_n/n).\qquad\square$$

Lemma 5.6. *Let* $f\in C(\mathbf{R})$ *be uniformly continuous on* $\mathbf{R}$. *Then there exists a polynomial* $P\in\Pi_n$ *such that for* $x\in\mathbf{R}$ *we have*

$$|f(x)-P(x)|W_{rQ}^{v}(x)\leqslant C\omega(f;\mathbf{R},a_n/n),\tag{5.1}$$

$$|P^{(j)}(x)|W_{rQ}^{v}(x)\leqslant C_j(n/a_n)^j\omega(f;\mathbf{R},a_n/n),\quad j=0,1,2,\dots,\tag{5.2}$$

where $W_{rQn,2\lambda}$ *is defined in* (2.1), *and* C, C_j *are constants.*

Proof. By Teljakovskii [Te] we have the following. For $g \in C[-1,1]$, there exists $T(x) \in \Pi_n$ such that

$$|g(t) - T(t)| \leqslant C\omega([-1,1], g; (1-t^2)^{1/2}/n),$$

$$(1-t^2)^{1/2}|T'(t)| \leqslant Cn\omega([-1,1], g; (1-t^2)^{1/2}/n),$$

where $\omega([-1,1], g; h)$ is the modulus of continuity for g on $[-1,1]$. Therefore, we see that for $|x| \leqslant Da_n$, $D > 1$

$$|f(x) - P(x)| \leqslant C\omega(f; [-2Da_n, 2Da_n], 2Da_n/n) \leqslant C\omega(f; \mathbf{R}, a_n/n) \tag{5.3}$$

$$|P'(x)| \leqslant C(n/a_n)\omega(f; [-2Da_n, 2Da_n], 2Da_n/n) \leqslant C(n/a_n)\omega(f; \mathbf{R}, a_n/n). \tag{5.4}$$

For $|x| \leqslant Da_n$ we see that $|P(x)|W_{rQ}^{1/2}(x)$ is bounded. Because from (5.3) and Proof of Lemma 5.5

$$|P(x)W_{rQ}^{1/2}(x)| \leqslant C\{|f(x)|W_{rQ}^{1/2}(x) + \omega(f; \mathbf{R}, a_n/n)W_{rQ}^{1/2}(x)\} \leqslant C.$$

Therefore, by the infinite–finite range inequality [KaS1, Theorem 1.1] we have, for $|x| \geqslant Da_n$,

$$|P(x)|W_{rQ}^{1/2}(x) \leqslant C||PW_{rQ}^{1/2}||_{L_\infty\{|x| \leqslant a_n\}} \leqslant C.$$

So for $|x| \geqslant Da_n$ we have

$$|P(x)|W_{rQ}^{v}(x) \leqslant CW_{rQ}^{v-1/2}(Da_n) \leqslant C\omega(f; \mathbf{R}, a_n/n). \tag{5.5}$$

Consequently, we have, by (5.3), (5.5) and Lemma 5.5,

$$|f(x) - P(x)|W_{rQ}^{v}(x) \leqslant C\omega(f; \mathbf{R}, a_n/n), \quad x \in \mathbf{R},$$

that is we obtain (5.1).

We have to show (5.2). By (5.4) and the infinite–finite range inequality we have, for $|x| \geqslant Da_n$,

$$|P'(x)|W_{rQ}^{v}(x) \leqslant C||P'W_{rQ}^{v}||_{L_\infty\{|x| \leqslant a_n\}} \leqslant C(n/a_n)\omega(f; \mathbf{R}, a_n/n).$$

So, noting (5.4) for $x \in \mathbf{R}$,

$$|P'(x)|W_{rQ}^{v}(x) \leqslant C(n/a_n)\omega(f; \mathbf{R}, a_n/n). \tag{5.6}$$

Consequently, repeating of the Markov inequality (Theorem 2.1), the inequality (5.6) means

$$|P^{(j)}(x)|W_{rQ}^{v}(x) \leqslant ||P^{(j)}W_{rQ}^{v}||_{L_\infty(\mathbf{R})} \leqslant C_j(n/a_n)^j\omega(f; \mathbf{R}, a_n/n), \quad j = 0, 1, 2, \ldots,$$

so (5.2) is shown. Consequently, the lemma is complete. □

Definition 5.7. We define $\Phi_n(x) = \max\{n^{-2/3}, 1 - |x|/a_n\}^{1/4}$.

We note that for some positive constants C,

$$CW_{rQ}(x) \leqslant (1+|x|)^{-\nu\eta/6} \leqslant C\Phi_n^\nu(x) \leqslant C. \tag{5.7}$$

In fact, the first inequality is easy to show. For the second inequality if $x \leqslant (1/2)a_n$, then it is trivial, and if $(1/2)a_n < x$, then we see $(1+|x|)^{-\eta/6} \leqslant Ca_n^{-\eta/6} \leqslant Cn^{-1/6} \leqslant \Phi_n(x)$.

Lemma 5.8. *We write some basic results.*

(i) *If n is odd, then we have*

$$|P_{n-1}(0)| \sim (n/a_n)^r a_n^{-1/2},$$

$$|P_n'(0)| \sim (n/a_n)^r n a_n^{-3/2} \quad \text{[KaS1, \textit{Theorem} 1.9](i).}$$

(ii) *Uniformly for* $2 \leqslant j \leqslant n$, $n = 2, 3, 4, \ldots$, *we have*

$$Ca_n/n \leqslant x_{j-1,n} - x_{jn},$$

especially for $|x_{jn}|, |x_{j-1,n}| \leqslant \eta a_n, 0 < \eta < 1$, *we see*

$$x_{j-1,n} - x_{jn} \sim a_n/n \quad \text{[KaS1, \textit{Theorem} 1.10].}$$

Sketch of proof for Theorem 5.1. We recall the definitions of Hermite–Fejér interpolation polynomials. For $f \in C(\mathbf{R})$ we define

$$L_n(\nu, f; x) = \sum_{k=1}^{n} f(x_{kn}) h_{kn}(\nu; x),$$

and define for $f \in C^{(\nu-1)}(\mathbf{R})$,

$$L_n(\nu-1, \nu, f; x) = \sum_{k=1}^{n} \sum_{s=0}^{\nu-1} f^{(s)}(x_{kn}) h_{skn}(\nu-1, \nu; x).$$

Let $f \in C(\mathbf{R})$, and let $P \in \Pi_n$ satisfy inequalities (5.1) and (5.2).

$$\begin{aligned}
&W_{rQ}^\nu(x)(1+|x|)^{-\nu\eta/6}|f(x) - L_n(\nu, f; x)| \\
&\quad \leqslant W_{rQ}^\nu(x)(1+|x|)^{-\nu\eta/6}\{|f(x) - P(x)| + |L_n(\nu, f-P; x)| \\
&\qquad + \sum_{k=1}^{n} \sum_{s=1}^{\nu-1} |P^{(s)}(x_{kn})||h_{skn}(\nu-1, \nu; x)|\}. \\
&\quad \leqslant W_{rQ}^\nu(x)(1+|x|)^{-\nu\eta/6}|f(x) - P(x)| + W_{rQ}^\nu(x)(1+|x|)^{-\nu\eta/6} \\
&\qquad \times \sum_{k=1}^{n} W_{rQ}^\nu(x_{kn})|f(x_{kn}) - P(x_{kn})| W_{rQ}^{-\nu}(x_{kn})|h_{kn}(\nu; x)|
\end{aligned}$$

$$
\begin{aligned}
&+\ W_{rQ}^{v}(x)(1+|x|)^{-v\eta/6}\sum_{k=1}^{n}\ \sum_{s=1}^{v-1}|P^{(s)}(x_{kn})||h_{skn}(v-1,v;x)| \\
&\leqslant C\omega(f;\mathbf{R},a_n/n)(1+|x|)^{-v\eta/6}\left\{1+W_{rQ}^{v}(x)\sum_{k=1}^{n}\ W_{rQ}^{-v}(x_{kn})|h_{kn}(v;x)|\right\} \\
&+\ W_{rQ}^{v}(x)(1+|x|)^{-v\eta/6}\sum_{k=1}^{n}\ \sum_{s=1}^{v-1}|P^{(s)}(x_{kn})||h_{skn}(v-1,v;x)|. \qquad (5.8)
\end{aligned}
$$

We estimate the Lebesgue constant

$$
W_{rQ}^{v}(x)(1+|x|)^{-v\eta/6}\sum_{k=1}^{n}\ W_{rQ}^{-v}(x_{kn})|h_{kn}(v;x)|, \qquad (5.9)
$$

and the sum

$$
W_{rQ}^{v}(x)(1+|x|)^{-v\eta/6}\sum_{k=1}^{n}\ \sum_{s=1}^{v-1}|P^{(s)}(x_{kn})h_{skn}(v-1,v;x)|. \qquad (5.10)
$$

First, we estimate (5.9). We use Lemmas 1.5(a),(d),(e), 5.8, Corollary 3.4 and (5.7).

$$
\begin{aligned}
&W_{rQ}^{v}(x)(1+|x|)^{-v\eta/6}\sum_{k=1}^{n}\ |h_{kn}(v;x)| \\
&\leqslant \sum_{x_{kn}\neq 0}\left|\frac{W_{rQ}(x)\Phi_n(x)P_n(x)}{(x-x_{kn})W_{rQ}(x_{kn})P_n'(x_{kn})}\right|^{v} \\
&\quad\times\sum_{i=0}^{v-1}|e_i(v,k,n)(x-x_{kn})^{i}| \\
&\leqslant \sum_{x_{kn}\neq 0}\left|\frac{W_{rQ}(x)\Phi_n(x)P_n(x)}{(x-x_{kn})\Phi_n^{-1}(x_{kn})W_{rQ}(x_{kn})P_n'(x_{kn})}\right|^{v} \\
&\quad\times\sum_{i=0}^{v-1}|e_i(v,k,n)(x-x_{kn})^{i}| \quad (\text{note } (5.7)) \\
&\leqslant \sum_{x_{kn}\neq 0}(1/j(x,k)) \\
&\leqslant C\log(1+n), \qquad (5.11)
\end{aligned}
$$

where

$$
|x-x_{kn}|\sim j(x,k)a_n/n.
$$

Next, we estimate (5.10). By above method and (5.2), we see

$$
\begin{aligned}
& W_{rQ}^{v}(x)(1+|x|)^{-v\eta/6} \sum_{k=1}^{n} \sum_{s=1}^{v-1} |P^{(s)}(x_{kn})||h_{skn}(v-1,v;x)| \\
&\quad = W_{rQ}^{v}(x)(1+|x|)^{-v\eta/6} \sum_{k=1}^{n} \sum_{s=1}^{v-1} |P^{(s)}(x_{kn}) W_{rQ}^{v}(x_{kn})| W_{rQ}^{-v}(x_{kn}) \\
&\qquad \times |h_{skn}(v-1,v;x)| \\
&\quad \leqslant \sum_{x_{kn}\neq 0} \sum_{s=1}^{v-1} \sum_{i=s}^{v-1} (n/a_n)^s \omega(f;\mathbf{R},a_n/n) \\
&\qquad \times \left| \frac{W_{rQ}(x)\Phi_n(x)P_n(x)}{(x-x_{kn})W_{rQ}(x_{kn})P_n'(x_{kn})} \right|^{v} |e_{si}(v,k,n)(x-x_{kn})^i| \\
&\quad \leqslant C\omega(f;\mathbf{R},a_n/n) \\
&\qquad \times \sum_{x_{kn}\neq 0} \sum_{s=1}^{v-1} \sum_{i=s}^{v-1} \left| \frac{W_{rQ}(x)\Phi_n(x)P_n(x)}{(x-x_{kn})\Phi_n^{-1}(x_{kn})W_{rQ}(x_{kn})P_n'(x_{kn})} \right|^{v} \\
&\qquad \times |(n/a_n)^s (n/a_n)^{i-s}(x-x_{kn})^i| \\
&\quad \leqslant C\omega(f;\mathbf{R},a_n/n) \sum_{x_{kn}\neq 0} (1/j(x,k)) \\
&\quad \leqslant C\log(1+n)\omega(f;\mathbf{R},a_n/n) \qquad (5.12)
\end{aligned}
$$

for n large enough.

Consequently, by (5.8), (5.11) and (5.12) the proof of the theorem is complete. □

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